ON INJECTIVE HOMOMORPHISMS BETWEEN TEICHMÜLLER MODULAR GROUPS

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ABSTRACT. In this paper we prove that injective homomorphisms between Teichmüller modular groups of compact orientable surfaces are necessary isomorphisms, if an appropriately measured "size" of the surfaces in question differs by at most one. In particular, we establish the co-Hopfian property for modular groups of surfaces of positive genus.

1. INTRODUCTION

Let S be a compact orientable surface. The *Teichmüller modular group* Mod_S of the surface S, also known as the *mapping class group* of S, is the group of isotopy classes of orientation preserving diffeomorphisms $S \to S$. The *pure modular group* PMod_S is the subgroup of Mod_S consisting of isotopy classes of diffeomorphisms which preserve each component of ∂S. The *extended modular group* Mod^*_S of S is the group of isotopy classes of *all* (including orientation-reversing) diffeomorphisms $S \to S$.

Before turning to the main results of the paper we would like to point out the following two theorems.

Theorem 1. *Let S be a compact connected orientable surface of positive genus. Suppose that S is not a torus with at most two holes. Then Mod_S is co-Hopfian, (i.e. every injective homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_S$ is an isomorphism).*

Note that Mod_S is also a Hopfian group, i.e. every surjective homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_S$ is an isomorphism. As is well known, a group is Hopfian if it is residually finite. The last property was proved for modular groups by E. Grossman [G]. See also [I3], Exercise 1.

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Theorem 1 provides an affirmative answer to a question communicated by D. D. Long to the first author: "Is every injective homomorphism $\rho : \text{Mod}_S \to \text{Mod}_S$ an isomorphism provided S is a closed surface of genus greater than 1?"

Note that it is usually quite nontrivial to establish the co-Hopfian property for a group of geometric interest. G. Prasad [P] proved that irreducible lattices in linear analytic semisimple groups are co-Hopfian if the dimension of the associated symmetric space is $\neq 2$ (cf. [P], Proposition). After the initial results of M. Gromov [GG] (cf. [GG], 5.4.B), E. Rips and Z. Sela [RS] and Z. Sela [S] proved recently the co-Hopfian property for a wide class of hyperbolic groups (cf. [RS], Section 3). So, Theorem 1 turns out to be a new instance of the well know but still mysterious analogy between modular groups and lattices and between Teichmüller spaces and hyperbolic spaces.

Theorem 2. *Let S be a closed orientable surface of genus at least 2. Then there is no injective homomorphisms $\text{Out}(\pi_1(S)) \to \text{Aut}(\pi_1(S))$. In particular, the natural epimorphism $\text{Aut}(\pi_1(S)) \to \text{Out}(\pi_1(S))$ is nonsplit.*

The second statement of Theorem 2 answers a question of J. S. Birman, stated as a part of Problem 8 of S. M. Gersten's list of Selected Problems in [GS]. For genus at least 3 this nonsplitting result follows also from the results of G. Mess [Me], as explained in the next paragraph. (The relation between the Mess' paper [Me] and the Birman's question apparently went unnoticed.) Birman asked whether the natural homomorphism $\text{Aut}(\pi_1(S)) \to \text{Out}(\pi_1(S))$ splits when S is a closed surface of genus greater than 1. Note that this homomorphism is an isomorphism when S is a sphere or a torus. Birman suggested this question as an algebraic variation of the generalized Nielsen realization problem, which also has an affirmative solution when S is a sphere or a torus. Contrary to the statement in Problem 8 in [GS], it is not equivalent to this realization problem. The realization problem seems to be of a different nature than Birman's question.

Theorems 1 and 2 are deduced from our main results, concerned with injective homomorphisms between Teichmüller modular groups. The relationship between Long's question and injective homomorphisms between modular groups needs no explanation. The relationship between Theorem 2 and injective homomorphisms between modular groups may be summarized as follows. Let S be a closed surface of genus greater than 1 and S' be the surface obtained from S by deleting the interior of a disc containing the basepoint x for the fundamental group $\pi_1(S)$. Every self-diffeomorphism of S' extends to a self-diffeomorphism of S fixing x. In this way, we obtain a natural homomorphism $\text{Mod}^*_{S'} \to \text{Mod}^*_S$. One may identify this homomorphism with $\text{Aut}(\pi_1(S)) \to \text{Out}(\pi_1(S))$; cf. the proof of Theorem 14.2 for details. Given this (fairly well known) identification, the above answer to Birman's question for genus at least 3 follows from the Proposition 2 of G. Mess' paper [Me]. And the more general result of Theorem 2 about the nonexistence of injective homomorphisms

$\text{Out}(\pi_1(S)) \to \text{Aut}(\pi_1(S))$ follows from the nonexistence of injective homomorphisms $\text{Mod}_S \to \text{Mod}_{S'}$.

The results of [I2] and [M] provide a complete description of automorphisms of modular groups. Roughly speaking, essentially all automorphisms of modular groups are geometric. Based upon these results on automorphisms, one might expect that essentially all injective homomorphisms between modular groups are geometric. Our main results, stated below as Theorems 3–6, verify this expectation for a large class of pairs (S, S').

Theorem 3. *Let S and S' be compact connected orientable surfaces. Suppose that the genus of S is at least 2 and S' is not a closed surface of genus 2. Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\text{Mod}_{S'}$ *differ by at most one. If $\rho : \text{Mod}_S \to \text{Mod}_{S'}$ is an injective homomorphism, then ρ is induced by a diffeomorphism $H : S \to S'$, (i.e. $\rho([G]) = [HGH^{-1}]$ for every orientation preserving diffeomorphism $G : S \to S$, where we denote by $[F]$ the isotopy class of a diffeomorphism F). In particular, ρ is an isomorphism.*

If we strengthen the hypothesis on the maxima of ranks of abelian subgroups, we can allow S to be of genus one also, with only few exceptions. The finitely many exceptional pairs of surfaces (S, S'), referred to in the following theorem, are listed in Section 10.4.

Theorem 4. *Let S and S' be compact connected orientable surfaces. Suppose that S has positive genus, S is not a torus with at most one hole, S' is not a closed surface of genus 2 and (S, S') is not an exceptional pair. If the maxima of ranks of abelian subgroups of* Mod_S *and* $\text{Mod}_{S'}$ *are equal and $\rho : \text{Mod}_S \to \text{Mod}_{S'}$ is an injective homomorphism, then ρ is induced by a diffeomorphism $S \to S'$.*

The maxima of ranks of abelian subgroups of Mod_S can be easily computed: it is equal to $3g - 3 + b$, where g is the genus and b is the number of boundary components of S, according [BLM]. This number turns out to be a convenient measure of "size" of a surface S.

Similar results are obtained when S' is a closed surface of genus 2. The statements involve the exceptional outer automorphism $\tau : \text{Mod}_{S'} \to \text{Mod}_{S'}$ which maps a Dehn twist about a nonseparating circle on S' to its product with the unique nontrivial element of the center of $\text{Mod}_{S'}$ ([I2], [M]). No restrictions on the maxima of the ranks of abelian subgroups are needed in the following theorem because, in fact, under its assumptions the maxima of ranks are automatically equal (cf. the proof of Theorem 12.16).

Theorem 5. *Let S be a compact connected orientable surface of genus at least 2. Let S' be a closed surface of genus 2. Let τ be the exceptional outer automorphism of* $\text{Mod}_{S'}$. *If $\rho : \text{Mod}_S \to \text{Mod}_{S'}$ is an injective homomorphism, then either ρ or $\tau \circ \rho$ is induced by a diffeomorphism $S \to S'$.*

Theorem 6. *Let S be a compact connected orientable surface of positive genus. Let S' be a closed surface of genus 2. Let τ be the exceptional automorphism of* $\mathrm{Mod}_{S'}$*. If the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *are equal and* $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ *is an injective homomorphism, then either ρ or $\tau \circ \rho$ is induced by a diffeomorphism $S \to S'$.*

Theorems 3–6 generalize the following results of [I2] and [M].

Corollary 1. ([I2]) *Let S be a compact connected orientable surface of positive genus. Suppose that S is not a torus with at most two holes or a closed surface of genus 2. Then every automorphism of* Mod_S *is given by the restriction of an inner automorphism of* Mod^*_S*. In particular,* $\mathrm{Out}(\mathrm{Mod}_S)$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$*.*

Corollary 2. ([M]) *Let S be a closed surface of genus 2. Let τ be the exceptional outer automorphism of* Mod_S*. Then every automorphism of* Mod_S *is given by the restriction of an inner automorphism of* Mod^*_S *or by the composition of such an automorphism with τ. In particular,* $\mathrm{Out}(\mathrm{Mod}_S)$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$*.*

Note that Theorem 1 follows immediately from Theorems 4 and 6 and, assuming the brief explanation given above, Theorem 2 follows from Theorems 3 and 5.

While Theorems 3–6 are, probably, not the best possible in terms of the restrictions on the pair (S, S'), some restrictions are certainly necessary in order to keep their conclusions. In fact, we will construct several families of examples of injective homomorphisms $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ such that Mod_S and $\mathrm{Mod}_{S'}$ are not isomorphic. Since our examples are still in a natural sense geometric, this leaves place for the hope that all injective homomorphisms between modular groups are geometric in an appropriate sense.

The techniques employed in this paper are geometric in nature. Like those employed in [BLM], [I2] and [M], they are based upon Thurston's theory of surface diffeomorphisms. More precisely, the arguments of this paper play upon restrictions upon commuting elements in Mod_S which follow from Thurston's theory. We say that an injective homomorphism is *twist-preserving* if it sends Dehn twists about nonseparating circles to Dehn twists. The crucial step in the proof of Theorems 3–6, as in the proof of Corollary 1 in [I2], is to show that an injective homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is twist-preserving. The last property forces an injective homomorphism to be induced by a diffeomorphism $S \to S'$, provided the genus of S is at least 2, without any additional assumptions on S'. (This crucial step fails when S' is a closed surface of genus 2. However, as in the proof of Corollary 2 in [M], the failure is exactly compensated for by the exceptional outer automorphism $\tau : \mathrm{Mod}_{S'} \to \mathrm{Mod}_{S'}$.) Since our homomorphisms are only injective, the reduction to twist-preserving homomorphisms does not follow immediately from the algebraic characterization of Dehn twists given in [I2]. We do not know of an algebraic characterization of Dehn twists which would yield an immediate reduction in the present context. Nevertheless, the assumption on the maxima of the ranks of abelian subgroups allows us to complete this crucial step of the argument. At the same time, under this assumption we are able to deal with the twist-preserving homomorphisms in the case when S has genus 1 also, with few exceptions.

Here is an outline of the paper. In Section 2, we review the basic notions and results related to Teichmüller modular groups. We assume that the reader is familiar with the fundamentals of Thurston's theory of surfaces (cf. [FLP]) and do *not* recall them. We also correct some minor mistakes in [I3]; cf. Remark 2.14.

In Section 3, we discuss two general constructions of injective homomorphisms between modular groups: (i) doubling and (ii) lifting to certain characteristic covers. These constructions produce examples which are not induced by diffeomorphisms, providing a contrast to our results on injective homomorphisms. We close the section with a family of examples with source a torus with one hole. These "hybrid" examples are produced by iterating modified versions of the doubling and lifting constructions. The results of this section are not used in the rest of the paper.

Section 4 discusses the basic relations between a pair of Dehn twists. The results of this section will play a role in the present context parallel to that played by Theorem 3.1 of [I2] and Lemma 4.3 of [M] in the original proofs of Corollaries 1 and 2 above. That is, they allow us to conclude that an injective homomorphism respects the geometric intersection properties of certain configurations of circles. These results are implicit in the proof of Lemma 4.3 of [M].

Section 5 concerns the relationship between Dehn twists supported on neighborhoods of boundary components of S and Dehn twists supported on nontrivial circles on S. The main tool in this section is the well known "lantern" relation. Roughly speaking, the results of this section allow us to conclude that an injective homomorphism respects the distinction between boundary components and nonseparating circles.

In Section 6, we discuss the centers of modular groups and closely related subgroups. The results of this section are well known. Since there does not seem to be an account of these results in the literature, we give complete proofs here. These results play the same role in the present context as in [I2] and [M]. That is, roughly speaking, they allow us to conclude that an injective homomorphism is controlled by the correspondence it induces between certain circles on the source and the target.

In Section 7 is devoted to a technical tool crucial for the next Section 8: a special configuration of circles on S and its basic properties. It will be used also in Section 13.

In Section 8 we prove, under two different (but overlapping) assumptions that any injective twist-preserving homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}'_S$ is, in fact, induced by some diffeomorphism $S \to S'$ (and, in particular, is an isomorphism). This is done in Theorem 8.9 under the assumption that the genus of S is at least 2, and in Theorem 8.15 under the assumption that the genus of S is at least 1 and the maxima of ranks of maximal abelian subgroups differ by at most 1. (This is the first place where we see the assumption on maxima of ranks of abelian subgroups entering into our arguments.) The first step in the proof is familiar from [I2] and [M]. Namely, by appealing to the results of Section 4, the configuration of circles on S introduced in Section 7 is shown to correspond to a configuration on S' with the same intersection properties. It is then shown that this correspondence of configurations of circles extends to an embedding $H : S \to S'$. If S is closed, then H must be a diffeomorphism. In fact, the embedding H must be a diffeomorphism under weaker hypotheses on S. In order to prove this, it suffices to show that H sends components of ∂S to components of $\partial S'$. Our arguments to this end employ the results of Section 5. Ultimately, the key role is played by the "lantern" relation.

Section 9 concerns systems of circles on S whose components are topologically equivalent on S. We prove that, except for a finite number of surfaces S, the components of such a system C must be nonseparating, provided the number of components of C differs by at most one from the number of components of a maximal system of circles on S. (This assumption, of course, is naturally related to the assumption on the maxima of ranks of abelian subgroups in Theorems 3–6.) Roughly speaking, these results allow us to conclude that an injective homomorphism respects the distinction between nonseparating and separating circles. The finite number of exceptions to the results of this section account for the surfaces S and S' explicitly excluded in the statement of Theorem 4.

We say that an injective homomorphism is *almost twist-preserving* if it sends some power of a Dehn twist about any nonseparating circle to a power of a Dehn twist. Section 10 is devoted to extending the results of Section 8 to almost twist-preserving injective homomorphisms.

The main task of the next three sections is to reduce the proofs of Theorems 3–6 to the case almost twist-preserving homomorphisms. In Section 11, we compute the centers of centralizers of mapping classes in pure subgroups of modular groups of finite index. These centers are free abelian groups of finite rank. Roughly speaking, these results will allow us to control the images of powers of Dehn twists under an injective homomorphism. Hence, they play a similar role in the present context as that played by the algebraic characterization of Dehn twists in [I2] and the "virtual" characterization of [M]. These results, however, do not provide a "virtual" characterization of Dehn twists. Hence, they do not lead immediately to the desired reduction.

In Section 12 we prove Theorems 1, 4, 5 and 6. On the basis of the results in Section 11, we are immediately led to conclude that the canonical reduction systems of the images of Dehn twists about nonseparating circles contain at most two components. The next step of the argument is to rule out pseudo-Anosov components of these images. Roughly speaking, this is done on the basis that there is not enough room in the target surface. At this point, we know that powers of these images are multitwists about one or two circles on the target surface. In the first case our homomorphism is almost twist-preserving. In the second case we derive several additional properties of it, which imply, in particular, that the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by one. It is then easy to see that this contradicts to the assumptions of Theorems 1, 4, 5 and 6.

Section 13 is devoted to the proof of Theorem 3. Here the maxima of the ranks of abelian subgroups can actually differ. The argument is based on a two fold proof by contradiction: (i) as we saw in Section 12, the contrary assertion implies the existence of a reduction circle, (ii) the lantern relation implies that some power of the Dehn twist along this reduction circle is in the image of the injective homomorphism. Since the Dehn twist along a reduction circle commutes with this image, condition (ii) contradicts the discussion in Section 5. In this way, we complete the proofs. In particular, the lantern relation is the key to the whole argument here. As an application of Theorem 3, we prove a nonsplitting result at the the end of this section; cf. Theorem 13.8.

Finally, Section 14 is devoted to the proof of Theorem 2.

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2. Preliminaries

In this section, we discuss background material used throughout the paper. We start with a review of basic definitions and notations used in the paper. Then, we turn to more special notions of a *pure element* and of a *reduction system.* We will not only review the definitions and the basic properties, but also extend some results about reduction systems proved in [I3] from elements acting trivially on $H(S, \mathbb{Z}/m_0\mathbb{Z})$ for some $m_0 \geq 3$ (which are always pure) to general pure elements. (To a big extent, these more general results are implicitly contained in [I3].) Working in such a generality has some advantages explained and illustrated by examples in Remark 2.13. The main results are summarized in Theorems 2.7–2.11. The discussion of this topic provided us with an opportunity to correct some arguments in [I3], which is done in Remark 2.14.

2.1. Notations and basic notions. Let S be a compact connected orientable surface of genus g with b boundary components. A *circle* on S is a one-dimensional closed connected submanifold of S. We recall that a circle on S is *nontrivial* if it is not the boundary of a disc in S and cannot be deformed into ∂S. By $V(S)$ we denote the set of isotopy classes of nontrivial circles on S. It is empty precisely when S is a sphere, a disc, an annulus or a disc with two holes. The extended modular group Mod_S^* of S acts in an obvious way on $V(S)$. Usually we are interested only in the action of modular groups Mod_S itself. By $V_0(S)$ we denote the subset of $V(S)$ consisting of isotopy classes of nonseparating circles on S. This subset is equal to $V(S)$ precisely when S is a sphere, a disc, an annulus, a disc with two holes or a torus with at most one hole. The action of Mod_S on $V(S)$ restricts to an action on $V_0(S)$. The group Mod_S also acts in an obvious way on the set of boundary components of S. The pure modular group PMod_S is nothing else as the kernel of this action, (i.e. $\mathrm{PMod}_S = \{f \in \mathrm{Mod}_S | f(c) = c$ for each component c of $\partial S\}$). Since every permutation of the components of ∂S is induced by an orientation preserving diffeomorphism $S \to S$, the group PMod_S is a normal subgroup of index $b!$ in Mod_S.

The most important elements of Mod_S are the *Dehn twists*. If S is *oriented*, we may distinguish between the *right* and *left* Dehn twists. If an orientation of the surface S is fixed, then, for each circle a on S, we denote by t_a the *right* Dehn twist *about* a; $t_a \in \mathrm{Mod}_S$. Then t_a^{-1} is the *left* Dehn twist about a. The Dehn twist t_a only depends upon the isotopy class α of a. This allows us also denote it by t_α and call it the *Dehn twist about* α. If a is a trivial circle on S, then t_a is the trivial element of Mod_S. According to Dehn [D], PMod_S is generated by Dehn twists along nontrivial circles on S. If $g \geq 1$, then, moreover, PMod_S is generated by Dehn twists along nonseparating circles on S. Cf., for example, [I1], Section 5.2 for an easy proof.

A one-dimensional submanifold C of S is called a *system of circles* on S, if the components of C are nontrivial and pairwise nonisotopic. If C is a system of circles, we call any composition of powers of Dehn twists about components of C a *multitwist* about C. In particular, any power of a Dehn twist is a multitwist.

If $g \geq 1$, then there exists a maximal system C of circles on S such that each component of C is a nonseparating circle on S. Moreover, any system of nonseparating circles on S may be enlarged to a maximal system of circles on S such that each component of C is a nonseparating circle on S. Any maximal system of circles on S consists of $\max\{0, 3g-3+b\}$ components, if S is not a closed torus (when it consists of one circle).

For every system of circles C on S, we denote by S_C the result of cutting S along C. Each component Q of S_C is a compact connected orientable surface. If S has negative Euler characteristic, then each component Q of S_C has negative Euler characteristic. Each component of ∂Q corresponds either to a component of ∂S or to a component of C. If at least one component of ∂Q corresponds to a component of ∂S, we say that Q is *peripheral* to S. Otherwise, we say that Q is *interior* to S. If no two components of ∂Q correspond to the same component of C, we say that Q is *embedded* in S. If S has negative Euler characteristic, then C is maximal if and only if each component of S_C is a disc with two holes. We say that two distinct components a and b of a system of circles are *adjacent* if there exists a component Q of S_C such that a and b both correspond to components of ∂Q.

For a system of circles C on S we denote by T_C the subgroup of Mod_S generated by the Dehn twists about the components of C. The group T_C is a free abelian and is freely generated by these Dehn twists. By the definition, elements of T_C are multitwists about C.If S has negative Euler characteristic, then by Theorem A of [BLM], any abelian subgroup G of Mod_S is finitely generated with torsion free rank bounded above by $3g-3+b$. Otherwise, the rank is bounded by 1 if S is a closed torus, and by 0 if S is sphere with ≤ 3 holes. Moreover, these bounds are exact. In particular, if C is a maximal system of circles on S, then T_C is a free abelian subgroup of maximum rank in Mod_S.

The *complex of curves* $C(S)$ of S is a simplicial complex on the set of vertices $V(S)$. A subset σ of $V(S)$ is a *simplex* of $C(S)$ if there exists a system of circles C on S such that σ is the set of isotopy classes of components of C. Any such system of circles C is called a *realization* of σ. A realization of a simplex σ is well defined up to isotopy on S. If t is a multitwist about a realization C of σ, we will say also that t is a *multitwist* about σ.

For every pair of simplices σ and τ of $C(S)$, the *geometric intersection number* $i(\sigma,\tau)$ of σ and τ is the minimum number of points of $C\cap D$ over all realizations C of σ and D of τ. The intersection number $i(\sigma,\tau)=0$ if and only if $\sigma\cup\tau$ is a simplex of $C(S)$. In particular, a subset σ of $V(S)$ is a simplex of $C(S)$ if and only if $i(\alpha,\beta)=0$ for every pair of vertices $\alpha,\beta\in\sigma$. If C and D are systems of circles on S, we denote by $i(C,D)$ the intersection number $i(\sigma,\tau)$, where σ is the simplex corresponding to C and τ is the simplex corresponding to D. We say that C and D are in *minimal position* if the number of points of intersection of C and D is equal to $i(C,D)$ and C is transverse to D. We say that a configuration (i.e. a set) of circles is in *minimal position* if each pair of circles of the configuration is in minimal position.

2.2. Pure elements. A diffeomorphism $F: S\to S$ is called *pure* if there is a system of circles C on S such that (F,C) satisfies the following condition:

Condition P. All points of C and ∂S are fixed by F, F_C preserves each component of S_C and, for each component Q of S_C, F_Q is isotopic either to a pseudo-Anosov diffeomorphism or to the identity.

This condition is the same as the Condition (P) of [I2] and is slightly stronger than the Condition (P) of [I3] in that in [I3] it is not required that F is fixed on ∂S. (In fact, in all applications of the Condition (P) in [I3] the more strong version is also fullfilled, as it follows from Theorem 1.2 of [I3].)

An element f of Mod_S is called *pure* if there is a pure diffeomorphism $F\in f$. If f is a pure element of Mod_S and $f^n(\alpha)=\alpha$ for a vertex α of $C(S)$ and some $n\neq 0$, then $f(\alpha)=\alpha$; cf. [I3], Corollary 3.7 (we can apply it because our Condition P is stronger than Condition (P) of [I3]). By Corollary 1.8 of [I3], there exists a subgroup Γ of finite index in Mod_S consisting entirely of pure elements. In particular, for any element $f\in\mathrm{Mod}_S$ some power $f^n, n\neq 0$ of it is pure. Note also that all Dehn twists (and all multitwists) are pure.

2.3. Reduction systems. The action of Mod_S^* on $V(S)$ extends to a simplicial action of Mod_S^* on $C(S)$. For the most of the paper, we are interested only in the induced action of Mod_S on $C(S)$. For each simplex σ of $C(S)$, we denote the stabilizer of σ in Mod_S by $M(\sigma)$. A simplex σ of $C(S)$ is a *reduction system* for $f \in \mathrm{Mod}_S$ if $f(\sigma) = \sigma$, (i.e. $f \in M(\sigma)$). If σ is a reduction system for $f \in \mathrm{Mod}_S$ and C is any realization of σ, then we may choose a diffeomorphism $F \in f$ such that $F(C) = C$. Any such diffeomorphism F determines a diffeomorphism $F_C : S_C \to S_C$ of the surface S cut along C. The isotopy class f_C of F_C depends only upon f. Hence, there is a natural *reduction homomorphism* $r_C : M(\sigma) \to \mathrm{Mod}_{S_C}$ given by the rule: $r_C(f) = f_C$. The kernel of r_C is equal to T_C.

Let σ be a simplex of $C(S)$, let C be a realization of σ and $R = S_C$. The group Mod_R acts in an obvious way on the set of components of R. For a component Q of R, we denote its stabilizer in Mod_R by $\mathrm{Mod}_R(Q)$ and the canonical *restriction homomorphism* $\mathrm{Mod}_R(Q) \to \mathrm{Mod}_Q$ by π_Q. If G is a subgroup of $M(\sigma)$ and Q is a component of S_C, we put $G_Q = \pi_Q(r_C(G) \cap \mathrm{Mod}_R(Q))$. Suppose that $f \in M(\sigma)$ and $f_C \in \mathrm{Mod}_R(Q)$. In this situation $F_C(Q) = Q$ for any $F \in f$ such that $F(C) = C$. For such an F, we denote the restriction of F_C to Q by F_Q, and we denote by $f_Q \in \mathrm{Mod}_Q$ the isotopy class of F_Q, (i.e. $f_Q = \pi_Q(f_C)$).

If $F : S \to S$ is a pure diffeomorphism, f is its isotopy class and C is a system of circles on S such that (F, C) satisfies Condition P, we say that C is a *pure reduction system* for F or f. Let Q be a component of S_C. If F_Q is isotopic to a pseudo-Anosov diffeomorphism of Q, we say that Q is a *pseudo-Anosov component of* S_C with respect to F or f. Otherwise, we say that Q is a *trivial component of* S_C with respect to F (or f).

A reduction system σ for $f \in \mathrm{Mod}_S$ is called *pure* if there exists a diffeomorphism $F \in f$ and a realization C of σ such that C is pure reduction system for F. Tautologically, an element having a pure reduction system is pure.

A vertex α of $C(S)$ is called an *essential reduction class* for a pure element f if (i) $f(\alpha) = \alpha$; (ii) if $i(\alpha, \beta) \neq 0$, then $f(\beta) \neq \beta$. Clearly, if α and β are two essential reduction classes, then $i(\alpha, \beta) = 0$. It follows that the set of all essential reduction classes for f is a symplex and, hence, is a reduction system. We call it the *canonical reduction system* for f and denote it $\sigma(f)$. Clearly, $\sigma(hfh^{-1}) = h(\sigma(f))$ for any $h \in \mathrm{Mod}_S$. Also, it is easy to see that $\sigma(t_\alpha^n) = \{\alpha\}$ for any $\alpha \in V(S)$ and $n \neq 0$.

In view of 2.2, if σ is a reduction system for a pure element f, then each vertex of σ is fixed by f.

Finally, recall that for any element $f \in \mathrm{Mod}_S$ some power $f^n, n \neq 0$ is pure according to 2.2, and define the *canonical reduction system* of an arbitrary element f of Mod_S as the canonical reduction system for some pure power of f. This definition depends only upon f, and not upon the power involved (cf. [I3], Section 7.4).

Lemma 2.4. *If f is a pure element, then the canonical reduction system $\sigma(f)$ for f*

is pure and is contained in any other pure reduction system.

Proof. Let F be a pure diffeomorphism representing f and C be a pure reduction system for F. After deleting some components from C we will get a minimal pure reduction system C' for F, i.e. such a reduction system that we cannot discard any component from C' without violating the Condition P. In the terminology of [I3] this is expressed by saying that C' does not have superfluous components. And, according to [I3], Section 7.19, $\sigma(f)$ is exactly the set of isotopy classes of components of C'. Clearly, this implies both statements of the lemma. □

Corollary 2.5. *If f is a pure element, then $\sigma(f)$ is empty precisely when f is either trivial or pseudo-Anosov.*

Lemma 2.6. *If τ is a reduction system for a pure element f, then $\sigma(f) \cup \tau$ is a pure reduction system for f.*

Proof. It follows from the definition of essential reduction classes that $i(\sigma(f), \tau) = 0$. Hence, $\sigma(f) \cup \tau$ is a simplex of $C(S)$ and is a reduction system for f. In order to see that it is a pure reduction systems, let us choose a realization C of $\sigma(f)$ and a diffeomorphism $F \in f$ as in the Condition P. We can choose a realization D of $\sigma(f) \cup \tau$ containing C. Clearly, any component of D is contained either in C or in a component Q of S_C such that F_Q is isotopic to the identity. This implies our assertion. □

Theorem 2.7. *Let f be a pure element of* Mod_S. *Let σ be a reduction system for f, C be a realization of σ, and $F \in f$ such that $F(C) = C$. Then F leaves each component of $C \cup \partial S$ invariant, preserves their orientations, preserves the orientation of S, and also leaves each component of $S \setminus C$ invariant. In particular, if $f(\sigma) = \sigma$ for some simplex σ, then f fixes all vertices of σ.*

Theorem 2.8. *Let f be a pure element of* Mod_S. *Then f is either trivial or of infinite order.*

Theorem 2.9. *Let f be a pure element of* Mod_S, *τ be a reduction system for f and C be a realization of τ. Suppose that Q is a component of S_C. Then $f_C \in \mathrm{Mod}_{S_C}(Q)$ and f_Q is a pure element of* Mod_Q.

Proof. Given Lemmas 2.4 and 2.6, these three theorems are immediate. □

Theorem 2.10. ([I3]) *Let Γ be a subgroup of* Mod_S *consisting of pure elements. If f is a pseudo-Anosov element of Γ, then its centralizer in Γ is an infinite cyclic group generated by a pseudo-Anosov element.*

Proof. Cf. [I3], Lemma 8.13. □

Theorem 2.11. ([I3]) *Let G be a subgroup of* Mod_S *consisting of pure elements. Then G either contains a free group with two generators or is a free abelian group of rank $\leq 3g-3+b$, where g is the genus of S and b is the number of components of the boundary of S.*

Proof. This is a minor variation on Theorem 8.9 of [I3]. The proof of Theorem 8.9 of [I3] appeals to several results which involve the hypothesis that certain subgroups of Mod_S act trivially on $H_1(S, \mathbb{Z}/m_0\mathbb{Z})$ for some integer $m_0 \geq 3$. This hypothesis can be replaced by the assumption that the relevant subgroups consist entirely of pure elements. After making this change in hypothesis in the relevant results, the following additional changes should be made. Appeals to Corollary 1.8 of [I3] should be eliminated, since we have now assumed the conclusion of Corollary 1.8. Appeals to Theorem 1.2 of [I3] should be replaced by appeals to Theorem 2.7. Appeals to Corollary 1.5 of [I3] should be replaced by appeals to Theorem 2.8. Appeals to Lemma 1.6 of [I3] should be replaced by appeals to Theorem 2.9. With these changes in the arguments of [I3], we prove the result. □

2.12. Reduction of subgroups. Let Γ be a subgroup of Mod_S consisting of pure elements. If C is a system of circles on S and σ is the corresponding simplex of $C(S)$, we put $\Gamma(C) = M(\sigma) \cap \Gamma$. If $f \in \Gamma(C)$, then $f_C \in \mathrm{Mod}_{S_C}(Q)$ in view of Theorem 2.9. Now, let G be a subgroup of $M(\sigma)$ consisting entirely of pure elements. Then $G(C) = G$ and, hence, $r_C(G) \subset \mathrm{Mod}_{S_C}(Q)$ for every componet Q of S_C. It follows that $G_Q = \pi_Q(r_C(G))$. Furthermore, by Theorem 2.9, G_Q consists entirely of pure elements of Mod_Q, and by Theorem 2.8, G_Q is torsion free. Obviously, $r_C(G)$ lies in the product of the groups G_Q over all components of S_C. (This product naturally lies in Mod_{S_C}. Indeed, the intersection of the stabilizers $\mathrm{Mod}_{S_C}(Q)$ over all components Q is naturally isomorphic to the product of the groups Mod_Q over all components Q.) In the above setting, the homomorphism $r_C|G : G \to \mathrm{Mod}_{S_C}$ will be the main tool for studying G. Note that its kernel is equal to $T_C \cap G$.

2.13. Remark. While the hypothesis that G acts trivially on $H_1(S, \mathbb{Z}/m_0\mathbb{Z})$ assures that all elements of G are pure, it is not preserved under reduction. In this paper, we prefer to work with the weaker assumption that G consist entirely of pure elements, which is preserved under reduction by Theorem 2.9. An example showing that the triviality of the action on the homology is not preserved under reduction may be constructed as follows.

Let S be a closed surface of genus 3. We may express S as the union of two disjoint embedded tori with one hole, P_1 and P_2, joined by a torus with two holes Q. Let $c_i = \partial P_i, i = 1, 2$ and $C = c_1 \cup c_2$. Then C is a system of circles on S, S_C consists of three components, P_1, P_2 and Q, and $\partial Q = C$. Let $a \cup b$ be a maximal system of nonseparating circles on Q. Since Q is a torus with two holes and a and b are nonseparating circles on Q, $a \cup b$ separates Q into two discs with two holes, Q_1 and

Q_2. We assume that these discs with two holes are labeled so that $\partial Q_i = c_i \cup a \cup b$. Then $a \cup b$ separates S into two tori with two holes, $P_1 \cup Q_1$ and $P_2 \cup Q_2$.

Each circle on S determines a homology class on S which is well defined up to sign. The previous properties imply that a and b determine the same homology class (up to sign) on S. Fix an orientation of S and let $F = T_a \circ T_b^{-1}$, where T_a and T_b are right twist maps supported on neighborhoods of a and b in Q. The action of T_a on $H_1(S)$ depends only upon the homology class of a up to sign. This implies that F acts trivially on $H_1(S)$ and, hence, on $H_1(S, \mathbb{Z}/m_0\mathbb{Z})$.

Clearly, $F(C) = C$ and $F_Q = T_a \circ T_b^{-1}$. Orient $a \cup b \cup c_1$ as the boundary of Q_1. Then $a + b + c_1 = 0 \in H_1(Q)$. We may choose a circle d on Q such that $i(a,d) = i(b,d) = 1$. Let $\langle\,,\rangle$ denote the homological intersection form on $H_1(Q)$. We may orient d so that $\langle a, d\rangle = 1$. Since $\langle c_1, d\rangle = 0$, the previous relations imply that $\langle b, d\rangle = -1$. By a well known formula for the action of a Dehn twist on homology, $T_a(d) = d + \langle a, d\rangle a = d + a \in H_1(Q)$ and $T_b(d) = d + \langle b, d\rangle b = d - b \in H_1(Q)$. Suppose that F_Q acts trivially on $H_1(Q, \mathbb{Z}/m_0\mathbb{Z})$. Then the relation $d + a = d - b$ must hold in $H_1(Q, \mathbb{Z}/m_0\mathbb{Z})$. This implies that $a + b = 0 \in H_1(Q, \mathbb{Z}/m_0\mathbb{Z})$. Since Q retracts onto $a \cup b \cup d$, $H_1(Q, \mathbb{Z}/m_0\mathbb{Z})$ is a free $\mathbb{Z}/m_0\mathbb{Z}$ module on a, b and d. Hence, the previous relation is impossible. This proves that F_Q acts nontrivially on homology.

In this example, the canonical reduction system for F is realized by $a \cup b$, not by C. An example in which C is a realization of the canonical reduction system for F can be obtained as follows. Since the action of T_a on $H_1(S)$ and $H_1(Q)$ depends only upon the homology class of a (up to sign) on Q, we may replace a by any circle on Q homologous to a on Q. Hence, we may assume that a and b are in minimal position and $a \cup b$ fills Q, (i.e. $Q \setminus (a \cup b)$ is a union of discs). By a well known construction of pseudo-Anosov maps described in [FLP], this assumption implies that F_Q is a pseudo-Anosov map. It follows that C is the canonical reduction system for F. The previous discussion shows that F acts trivially on $H_1(S, \mathbb{Z}/m_0\mathbb{Z})$, but F_Q does not act trivially on $H_1(Q, \mathbb{Z}/m_0\mathbb{Z})$.

2.14. Remark. Throughout [I3] the condition that elements of some subgroup of Mod_S act trivially on $H(S, \mathbb{Z}/m_0\mathbb{Z})$ for some $m_0 \geq 3$ was used. This condition implies that these elements are pure and is the only known general condition implying this. But, the fact that it is not preserved under reduction causes some difficulties, as explained in Remark 2.13. These difficulties cannot be avoided completely, because they occur already in the proof of fact that the elements acting trivially on $H(S, \mathbb{Z}/m_0\mathbb{Z})$ for some $m_0 \geq 3$ are pure (and this result is needed to get a subgroup of finite index consisiting of pure elements). In [I3], this problem is taken care of in (the proof of) Lemma 1.6. But, at some other places in [I3], these difficulties led to slightly incorrect arguments. A general way to correct them is to use the Theorems 2.7–2.11 proved above. But they can be also corrected by making only few small changes in [I3], as we explain now. All references below in this Remark are to [I3].

In the proof of Theorem 5.9 at the end of the first paragraph on the p.53, the elements f_1, g_1 are, indeed, pure, as it follows from the above, but the argument given for this is incorrect. Instead of proving that they are pure, we can replace the elements f, g in the proof by their powers f^a, g^b, such that the new f_1, g_1 will be contained in $\Gamma_{S_E}(m_0)$ (if f, g are replaced by f^a, g^b, then f_1, g_1 are replaced by f_1^a, g_1^b respectively).

The Corollary 7.18 is correct as stated, but a slightly different version of it is more convenient for Section 8. (In Sections 9 and 10 it should be used as stated.) Namely, $\Gamma' = \rho_C(G) \cap \Gamma_R(m_0)$ can be replaced by the new $\Gamma' = \rho_C(G \cap \Gamma_S(m_0))$. The proof goes almost unchanged. We only need to note that $\Gamma' \subset \text{Mod}_R(Q)$ by Theorem 1.2 and replace the appeal to Corollary 1.5 by the appeal to Lemma 1.6. (The group G' disappears from the statement and the proof; alternatively, we may put $G' = \Gamma'$.)

In Corollary 8.5 and in Lemma 5.10 used in its proof one can replace the condition that the subgroup in question consisits of pure elements by the weaker condition that the subgroup is torsion free. Only this weaker condition is actually used in the proofs.

In the first paragraph of the proof of Lemma 8.7 the new version of Corollary 7.18 should be used. After this, one should notice that each group $r_Q(\Gamma')$ is torsion free by Lemma 1.6 and then apply the new version of Corollary 8.5. The rest of the proof of Lemma 8.7 remains unchanged.

In the proof of Theorem 8.9 again the new version of Corollary 7.18 should be used. Then, one should notice that the group $r_Q(\Gamma')$ is torsion free by Lemma 1.6, and, hence, it is an infinite cyclic group (generated by a pseudo-Anosov element) if and only if it contains an infinite cyclic group as a subgroup of finite index, in view of the new version of Corollary 8.5. Given this, one should refer to Corollary 7.15 instead of Theorem 5.12. The rest of the proof remains unchanged.

3. Nongeometric injective homomorphisms

Our purpose, in this section, is to give examples of injective homomorphisms $\rho : \text{Mod}_S \to \text{Mod}_{S'}$ between modular groups of compact connected orientable surfaces S and S' which are not induced by a diffeomorphism $S \to S'$. The results of this section are not used in the rest of the paper.

3.1. Doubling. A simple construction of such a homomorphism is provided by the classical topological construction of doubling a surface. Let S be a compact connected orientable surface with nonempty boundary and dS be the double of S. The double dS is the union of two copies of S, S and $-S$, meeting along their common boundary ∂S. The boundary ∂S is a system of circles on dS and the surface R obtained by cutting dS along ∂S is the disjoint union of two copies of S. Given any diffeomorphism $F : S \to S$, we can form the double $dF : dS \to dS$ of F, which is the unique diffeomorphism $dS \to dS$ which preserves ∂S and restricts to F on each of the two copies of S in dS. Clearly, $d(F \circ G) = dF \circ dG$. This doubling construction also applies to isotopies on S. Hence, there is a *doubling homomorphism* $\delta : \mathrm{Mod}_S \to \mathrm{Mod}_{dS}$ given by the rule $\delta([F]) = [dF]$, where we denote by $[H]$ the isotopy class of a diffeomorphism H.

Let σ be the simplex of $C(dS)$ corresponding to the common boundary ∂S of the two copies of S in dS. Since $dF(\partial S) = \partial S$ for every diffeomorphism $F : S \to S$, this simplex σ is a reduction system for $\delta(\mathrm{Mod}_S)$. Hence, we may consider the composition of δ with the reduction homomorphism $r_{\partial S} : M(\sigma) \to Mod(R)$. Since dF preserves S, we may also compose with the restriction homomorphism $\pi_S : \mathrm{Mod}_R(S) \to \mathrm{Mod}_S$. Since dF restricts to F on S, we conclude that $\pi_S r_C \delta$ is equal to the identity on Mod_S. Hence, $\delta : \mathrm{Mod}_S \to \mathrm{Mod}_{dS}$ is an injective homomorphism. Since S is not diffeomorphic to dS, the homomorphism δ is not induced by a diffeomorphism $S \to dS$.

This doubling construction can be modified as follows. Let C be a submanifold of ∂S and $d_C(S)$ be the double of S along C, which is, by definition, the union of two copies of S meeting along C. Let σ be the simplex of $C(d_S(S))$ realized by C. Recall that $M(\sigma)$ is the stabilizer of σ in Mod_S. Modifying the previous discussion, we obtain an injective homomorphism $\delta_C : M(\sigma) \to \mathrm{Mod}_{S'}$, where $S' = d_C(S)$. Since doubling S along C doubles the Euler characteristic of S, $d_C(S)$ is not diffeomorphic to S, unless S is an annulus and C is one component of ∂S. In this case, $M(\sigma)$ is trivial and, hence, δ_C is induced by any diffeomorphism $S \to d_C(S)$. In all other cases, δ_C is not induced by a diffeomorphism.

3.2. Lifting. A second construction is provided by lifting to characteristic covers. Recall that a cover $X^\sim \to X$ is called *characteristic* if the image of the fundamental group $\pi_1(X^\sim)$ in $\pi_1(X)$ is a characteristic subgroup, i.e. a subgroup invariant under all automorphisms of $\pi_1(X)$. For this construction, we consider a compact connected orientable surface S of genus $g \geq 1$ with one boundary component. Let R be the closed surface of genus g obtained by attaching a disc D to the boundary of S. Choose a point p in the interior of D. Let $\pi : R' \to R$ be a characteristic cover of index $n \geq 2$ and let $p' \in \pi^{-1}(p)$. Note that S is naturally embedded in R. Let $S' = \pi^{-1}(S)$. The covering π restricts to a covering $\pi|S' : S' \to S$ of index n. The surface S' is a compact connected orientable surface of genus $g' = ng - n + 1$ with n boundary components.

Suppose that $F : S \to S$ is a diffeomorphism. We may extend F to a diffeomorphism $G : (R, p) \to (R, p)$. Since $\pi : R' \to R$ is a characteristic cover, G lifts to a unique diffeomorphism $G' : (R', p') \to (R', p')$. Since G preserves S, the diffeomorphism G' restricts to a lift $F' : S' \to S'$ of F. Since F' does not depend upon the choice of extension G of F, we have defined a unique lift F' of F. In the same way, we may define a unique lift of any isotopy of S. Hence, we have a *lifting homomorphism* $\lambda : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$. Suppose that $F' : S' \to S'$ is isotopic to the identity map. Since

F' is fiber-preserving with respect to $\pi|S' : S' \to S$, Theorem 1 of [BH] implies that F is isotopic to the identity. Hence, $\lambda : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective homomorphism. Since the Euler characteristic of S is multiplied by n under a covering of index n, S is not diffeomorphic to S'. Hence, λ is not induced by a diffeomorphism.

This lifting construction can be modified as follows. Let S be a compact connected orientable surface S with at least one boundary component. Suppose that S is not an annulus. Let c be a component of ∂S, $M(c)$ be the stabilizer in Mod_S of c and R be the surface obtained by attaching a disc D to c. Choose a point p in the interior of D. As before, let $\pi : R' \to R$ be a characteristic cover of index $n \geq 2$, $p' \in \pi^{-1}(p)$ and $S' = \pi^{-1}(S)$. Modifying the previous discussion, we obtain an injective homomorphism $\lambda_c : M(c) \to \mathrm{Mod}_{S'}$. Again, an Euler characteristic argument implies that S' is not diffeomorphic to S. Hence, λ_c is not induced by a diffeomorphism.

3.3. Hybrids. As in 3.2, let S be a compact connected orientable surface S of genus $g \geq 1$ with one boundary component. Let $\lambda : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ be constructed by the lifting construction of 3.2. Since $n \geq 2$, S' has at least two boundary components. Hence, we may compose $\lambda : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ with the doubling homomorphism $\delta : \mathrm{Mod}_{S'} \to \mathrm{Mod}_{dS'}$ to obtain an injective homomorphism $\nu : \mathrm{Mod}_S \to \mathrm{Mod}_{dS'}$. Since S is not diffeomorphic to dS', ν is not induced by a diffeomorphism.

We may modify this two stage construction as follows. In our scheme for choosing a lift $F' : S' \to S'$ of a diffeomorphism $F : S \to S$, we have implicitly singled out a special component c of $\partial S'$. Let C be the complement of c in $\partial S'$. Let $M(c)$ and $M(C)$ be the stabilizers in Mod_S of c and C respectively. The image of $\lambda : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is contained in the stabilizers $M(c)$ and $M(C)$. Hence, we can compose λ with either of the doubling homomorphisms δ_c or δ_C obtained by doubling S' along c or C or with a lifting homomorphism λ_c associated to a characteristic cover of the surface obtained by attaching a disc to S' along c. Again, an Euler characteristic argument shows that these composite homomorphisms are not induced by diffeomorphisms.

Since the images of these composite homomorphisms stabilize various proper submanifolds, including particular components, of their respective target surfaces, these doubling and lifting constructions can be iterated in a variety of ways. Again, an Euler characteristic argument shows that these homomorphisms are not induced by diffeomorphisms. If desired, at any stage of this iteration, we can double along the entire boundary of the target surface, and terminate the process with an injective homomorphism from Mod_S into the modular group of a closed surface.

3.4. Tori. Let S be a torus with one hole. The surface R obtained by attaching a disc D to ∂S, as in subsection 3.2, is a torus. It is well known that Mod_R is isomorphic to $SL_2(\mathbb{Z})$. The natural homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_R$ is an isomorphism. Hence, there is a natural correspondence between injective homomorphisms $\mathrm{Mod}_S \to \mathrm{Mod}_R$ and $\mathrm{Mod}_{S'} \to \mathrm{Mod}_R$ for any surface R.

Doubling S gives us an injective homomorphism $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$, where S' is a closed surface of genus 2.

The characteristic covers R' of R correspond to the subgroups $m(\mathbb{Z}^2)$ of index m^2 in the fundamental group $\mathbb{Z}^2$ of the torus R. Hence, lifting gives us injective homomorphisms $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$, where S' is a torus with m^2 holes. Iterating the doubling process on S', we obtain injective homomorphisms $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S''}$ where S'' has Euler characteristic $-2^k m^2$. If we have terminated the process by doubling along an entire boundary, then S'' is a closed surface of genus $2^{k-1}m^2 + 1$.

Hence, we obtain injective homomorphisms $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ where S' is any closed surface of genus g', provided $g' = 2^l m^2 + 1$ for some nonnegative integers l and m. If g' is a number of this form with m positive and even, then there are at least two solutions to the equation $g' = 2^l m^2 + 1$. Hence, in this situation, we expect that there are essentially distinct injective homomorphisms $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$.

If we bring lifting into the iteration, we will broaden the range of target surfaces for injective homomorphisms from Mod_S. On the other hand, it is clear that the doubling and lifting iteration scheme will not produce closed targets of arbitrary genus. In particular, the Euler characteristics of all closed targets produced by this iteration scheme (other than S, R and the closed surface of genus 2) are divisible by a perfect square m^2 for some integer $m > 1$.

4. Relations between Dehn twists

In this section, S denotes a compact *oriented* surface. Our purpose, in this section, is to discuss basic relations between Dehn twists along circles on S. Recall that for each isotopy class $\alpha \in V(S)$, we denote by $t_\alpha \in \mathrm{Mod}_S$ the right Dehn twist about any circle $a \in \alpha$. Thus, t_α^{-1} is the left Dehn twist about a. It is well known that $f t_\alpha f^{-1} = t_{f(\alpha)}$ for any $f \in \mathrm{Mod}_S$, $\alpha \in V(S)$. Also, if $f \in \mathrm{Mod}_S^* \setminus \mathrm{Mod}_S$ (i.e., if f is orientation-reversing), then $f t_\alpha f^{-1} = t_{f(\alpha)}^*$ for any $\alpha \in V(S)$.

For each $\alpha \in V(S)$, t_α is a pure, reducible element of infinite order in Mod_S. Moreover, α is the canonical reduction system for t_α and its powers. These facts imply the following well-known result.

Theorem 4.1. *Let t_α, t_β be two right twists. Let j, k be two nonzero integers. Then $t_\alpha^j = t_\beta^k$ if and only if $\alpha = \beta$ and $j = k$.*

Theorem 4.2. *Let t_α, t_β be distinct right twists. Let j, k be two nonzero integers. Then:*

(i) $t_\alpha^j t_\beta^k = t_\beta^k t_\alpha^j$ if and only if $i(\alpha, \beta) = 0$,

(ii) $t_\alpha^j t_\beta^k t_\alpha^j = t_\beta^k t_\alpha^j t_\beta^k$ if and only if $j = k = \pm 1$ and $i(\alpha, \beta) = 1$.

Proof. This theorem generalizes Theorem 3.1 of [I2] and Lemma 4.3 of [M].

The "if" clause of (i) is trivial and the "if" clause of (ii) is well known and easy to check.

We shall deduce the "only if" clause of both assertions from the following special case of Proposition 1 from Exposé 4, Appendice of [FLP]:

$$i(t_\alpha^j(\beta),\beta)=|j|(i(\alpha,\beta))^2. \tag{4.1}$$

Our proof follows that of Lemma 4.3 of [M].

Suppose that $t_\alpha^j t_\beta^k = t_\beta^k t_\alpha^j$. This implies that $t_\alpha^j t_\beta^k t_\alpha^{-j} = t_\beta^k$. Equivalently, $t_{t_\alpha^j(\beta)}^k = t_\beta^k$. By Theorem 4.1, we conclude that $t_\alpha^j(\beta)=\beta$. Hence, by equation (4.1):

$$0=i(\beta,\beta)=i(t_\alpha^j(\beta),\beta)=|j|(i(\alpha,\beta))^2. \tag{4.2}$$

Since j is nonzero, we conclude that $i(\alpha,\beta)=0$.

Suppose that $t_\alpha^j t_\beta^k t_\alpha^j = t_\beta^k t_\alpha^j t_\beta^k$. Suppose that $i(\alpha,\beta)=0$. Then $t_\alpha^j t_\beta^k = t_\beta^k t_\alpha^j$. From the relation $t_\alpha^j t_\beta^k t_\alpha^j = t_\beta^k t_\alpha^j t_\beta^k$, we conclude that $t_\alpha^j = t_\beta^k$. Since α and β are distinct, this contradicts Theorem 4.1. Hence, $i(\alpha,\beta)\neq 0$.

Let $f = t_\alpha^j t_\beta^k$. Then $f t_\alpha^j f^{-1} = t_\beta^k$. Equivalently, $t_{f(\alpha)}^j = t_\beta^k$. By Theorem 4.1, we conclude that $f(\alpha)=\beta$ and $j=k$. Thus $t_\alpha^j t_\beta^k(\alpha)=\beta$. Hence:

$$i(t_\alpha^j(\beta),\beta) = i(t_\alpha^j(\beta),t_\alpha^j t_\beta^k(\alpha)) = i(\beta,t_\beta^k(\alpha)) = i(t_\beta^{-k}\beta,\alpha) = i(\beta,\alpha). \tag{4.3}$$

By equation (4.1), $i(t_\alpha^j(\beta),\beta)=|j|(i(\alpha,\beta))^2$. Thus $|j|(i(\alpha,\beta))^2=i(\alpha,\beta)$. Since $i(\alpha,\beta)\neq 0$, we conclude that $|j|=i(\alpha,\beta)=1$. Since we already know that $j=k$, this completes the proof. □

Theorem 4.3. *Let a and b be two circles on S intersecting transversely at one point, and let U be a neigborhood of $a\cup b$ diffeomorphic to a torus with one hole. Let c be the boundary circle of U. Then*

$$(t_a t_b)^6 = t_c.$$

Moreover, if T_a, T_b and T_c are twist maps representing t_a, t_b and t_c respectively and supported in U, then $(T_a\circ T_b)^6$ is isotopic to T_c by an isotopy supported in U.

Proof. This is well known and is essentially due to Dehn [D]. In order to give the due to this pioneer paper of Dehn, we indicate how to deduce this result from [D].

In §4 a) Dehn introduces two elements Δ_a,Δ_b, which correspond to t_a, t_b^{-1} respectively in our notations. In §5 c) he introduces elements $\Sigma=\Delta_a\Delta_b^{-1}$ and $T=\Delta_a^{-1}\Delta_b\Delta_a^{-1}$ and in §6 c) he proves that $\Sigma^3T^2=1$ and that T^4 is equal to t_c^{-1} in our notation. (Dehn works on the torus with one hole, which is, clearly, sufficient. He proves the relation $\Sigma^3T^2=1$ first in §5 d) in the situation when the boundary is allowed to be moved during the isotopies and then notices in §6 c) that the proof works for the fixed boundary also.) Now, in our notations, $\Sigma=t_at_b$ and

$T = t_a^{-1} t_b^{-1} t_a^{-1}$. Hence, $(t_a t_b)^6 = \Sigma^6 = (\Sigma^3)^2 = (T^{-2})^2 = T^{-4} = (T^4)^{-1} = t_c$. The theorem follows.

Note that the relation $\Sigma^3 T^2 = 1$ or, what is the same, $\Sigma^3 = T^{-2}$ means in our notations that $t_a t_b t_a t_b t_a t_b = t_a t_b t_a t_a t_b t_a$ and thus is equivalent to the relation $t_b t_a t_b = t_a t_b t_a$. □

5. Peripheral Twists

In this section, S denotes a compact *oriented* surface. For a circle a on S we denote by T_a a standard twist map supported on a neighborhood of a in S. So, T_a represents $t_a \in \mathrm{Mod}_S$. If a is a trivial circle on S, then t_a is the trivial element of Mod_S and so T_a represents the trivial element. Nevertheless, twist maps supported on neighborhoods of boundary components of S play a role in the arguments of this paper. In this section, we develop a relationship between these *peripheral* twists and twists along nontrivial circles on S.

Let $\mathcal{M}_S$ denote the group of orientation preserving diffeomorphisms $S \to S$ which fix ∂S pointwise modulo isotopies which fix ∂S pointwise. Let a be a nontrivial circle on S or a component of ∂S. Let $\tilde{t}_a$ denote the (isotopy) class of T_a in $\mathcal{M}_S$. Naturally, we call it the *Dehn twist about a in $\mathcal{M}_S$*. There is a natural homomorphism $\mathcal{M}_S \to \mathrm{PMod}_S$. This homomorphism is surjective and its kernel is equal to the group $\tilde{T}_{\partial S}$ generated by the Dehn twists $\tilde{t}_a$ about the components a of ∂S. These Dehn twists freely generate $\tilde{T}_{\partial S}$. Thus, $\tilde{T}_{\partial S}$ is a free abelian group of rank b. The natural homomorphism $\mathcal{M}_S \to \mathrm{PMod}_S$ maps $\tilde{t}_a$ to t_a.

Theorem 5.1. *Let S be a compact connected orientable surface. Suppose that $\mathcal{C}$ is a collection of nonseparating circles on S such that* PMod_S *is generated by the Dehn twists t_c along the circles c of $\mathcal{C}$. Then $\mathcal{M}_S$ is generated by the Dehn twists $\tilde{t}_c$ along the circles c of $\mathcal{C}$ and $\tilde{T}_{\partial S}$. Moreover, $\tilde{T}_{\partial S}$ is a central subgroup of $\mathcal{M}_S$.*

Proof. Let $\tilde{G}$ be the subgroup of $\mathcal{M}_S$ generated by the Dehn twists $\tilde{t}_c$ along the circles c of $\mathcal{C}$. Since the natural homomorphism $\mathcal{M}_S \to \mathrm{PMod}_S$ sends $\tilde{t}_c$ to t_c and has kernel $\tilde{T}_{\partial S}$, $\mathcal{M}_S$ is generated by $\tilde{G}$ and $\tilde{T}_{\partial S}$. Clearly, $\tilde{T}_{\partial S}$ is central in $\mathcal{M}_S$. This completes the proof. □

5.2. Lantern relation. Let us recall the well known "lantern" relation discovered by M. Dehn [D] (cf. [D] §7 g) 1)) and rediscovered and popularized by Johnson [J]. Let S_0 be a sphere with four holes. Label the boundary components of S_0 by $C_0, .., C_3$ and write T_i for a standard twist map supported on a neighborhood of C_i in S_0. For $1 \le i < j \le 3$, let C_{ij} denote a circle encircling C_i and C_j as in Figure 5.1. Let T_{ij} denote a standard twist map supported on a neighborhood of C_{ij} in S_0. Then $T_0 \circ T_1 \circ T_2 \circ T_3$ is isotopic to $T_{12} \circ T_{13} \circ T_{23}$ by an isotopy which is fixed on ∂S_0.

Suppose that S_0 is embedded in S. Diffeomorphisms T_i and T_{ij} may be extended by the identity to all of S. In this sense, we may regard T_i and T_{ij} as standard twist

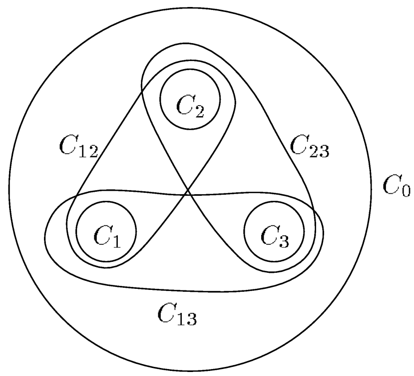


Figure 5.1

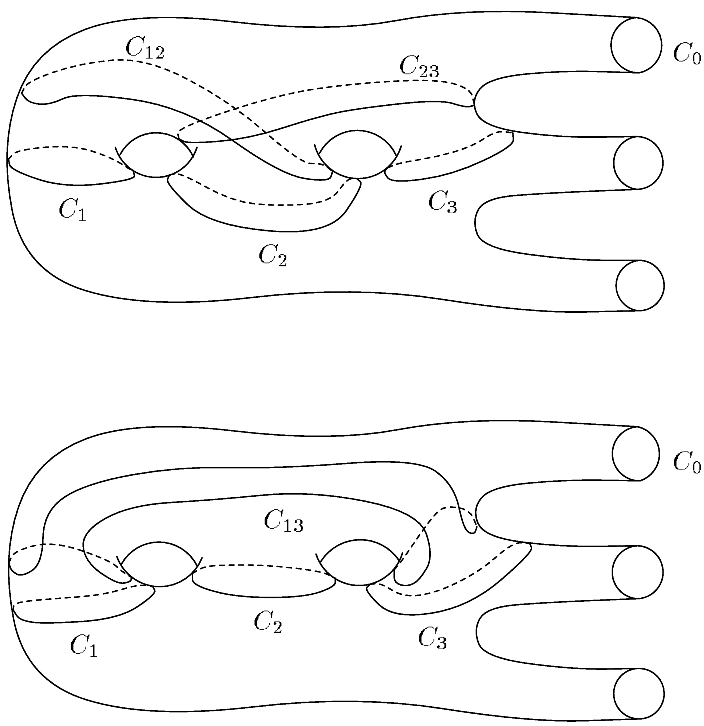


Figure 5.2

maps on S supported on neighborhoods of circles C_i and C_{ij}. Let $\tilde{t}_i \in \mathcal{M}_S$ denote the Dehn twist $\tilde{t}_{C_i}$ and $\tilde{t}_{ij} \in \mathcal{M}_S$ denote the Dehn twist $\tilde{t}_{C_{ij}}$. Any isotopy on S_0 which is fixed on ∂S_0, extends by the identity to all of S. Hence, the above discussion provides a relation in $\mathcal{M}_S$:

$$\tilde{t}_0\tilde{t}_1\tilde{t}_2\tilde{t}_3 = \tilde{t}_{12}\tilde{t}_{13}\tilde{t}_{23}. \tag{5.1}$$

Theorem 5.3. *Let S be a compact connected orientable surface of genus $g \geq 2$. Let $\mathcal{C}$ be a collection of nonseparating circles on S such that* PMod_S *is generated by the Dehn twists t_c along the circles c of $\mathcal{C}$. Then $\mathcal{M}_S$ is generated by the Dehn twists $\tilde{t}_c$ along the circles c of $\mathcal{C}$. Moreover, $\tilde{T}_{\partial S}$ is contained in the commutator subgroup of $\mathcal{M}_S$.*

Proof. Let $\tilde{G}$ be the subgroup of $\mathcal{M}_S$ generated by the Dehn twists $\tilde{t}_c$ along the circles c of $\mathcal{C}$. By Theorem 5.1, $\mathcal{M}_S$ is generated by $\tilde{G}$ and $\tilde{T}_{\partial S}$. It suffices to show that $\tilde{t}_a \in \tilde{G}$ for each component a of ∂S. Note that, since $\tilde{T}_{\partial S}$ is central in $\mathcal{M}_S$, $\tilde{G}$ is a normal subgroup of $\mathcal{M}_S$.

Let a be a component of ∂S. Since $g \geq 2$, we may embed S_0 in S so that: (i) $C_0 = a$; (ii) C_i is nonseparating on S for $1 \leq i \leq 3$; (iii) C_{ij} is nonseparating on S for $1 \leq i < j \leq 3$; cf. Figure 5.2. Let $\tilde{t}_i$ denote the Dehn twist along C_i in $\mathcal{M}_S$ and $\tilde{t}_{ij}$ denote the Dehn twist along C_{ij} in $\mathcal{M}_S$. Since C_i for $1 \leq i \leq 3$ and all C_{ij} are nonseparating, $\tilde{t}_i$ for $1 \leq i \leq 3$ and all $\tilde{t}_{ij}$ are conjugate in $\mathcal{M}_S$ to $\tilde{t}_1$. Hence, equation 5.1 implies that $\tilde{t}_0$ is equal to 0 in $H_1(\mathcal{M}_S)$ and, hence, $\tilde{t}_0$ is contained in the commutator subgroup of $\mathcal{M}_S$. Since $\tilde{t}_0 = \tilde{t}_a$, this implies that $\tilde{T}_{\partial S}$ is contained in the commutator subgroup of $\mathcal{M}_S$.

Since $\tilde{G}$ and $\tilde{T}_{\partial S}$ generate $\mathcal{M}_S$, we may choose an element $\tilde{g}_1 \in \tilde{G}$ and an element $\tilde{t} \in \tilde{T}_{\partial S}$ such that $\tilde{t}_1 = \tilde{g}_1\tilde{t}$. Since $\tilde{G}$ is a normal subgroup of $\mathcal{M}_S$ and $\tilde{t}$ is a central element of $\mathcal{M}_S$, we conclude that there exists elements $\tilde{g}_i$ and $\tilde{g}_{ij}$ of $\tilde{G}$ such that:

$$\tilde{t}_i = \tilde{g}_i\tilde{t} \ \text{ for } \ 1 \leq i \leq 3\,; \ \ \tilde{t}_{ij} = \tilde{g}_{ij}\tilde{t} \tag{5.2}$$

(recall that $\tilde{t}_i$ for $1 \leq i \leq 3$ and all $\tilde{t}_{ij}$ are conjugate to $\tilde{t}_1$). Since $\tilde{t}$ is a central element of $\mathcal{M}_S$, equations (5.1) and (5.2) imply that:

$$\tilde{t}_0\tilde{g}_1\tilde{g}_2\tilde{g}_3 = \tilde{g}_{12}\tilde{g}_{13}\tilde{g}_{23}. \tag{5.3}$$

Since $\tilde{g}_i$ and $\tilde{g}_{ij}$ are elements of $\tilde{G}$, equation (5.3) implies that $\tilde{t}_0 \in \tilde{G}$. Since $\tilde{t}_0 = \tilde{t}_a$, this completes the proof. □

6. Centers of Modular Groups

In this section, S is a compact connected orientable surface. Our goal is to describe the centralizers of PMod_S in Mod_S. The main results are summarized in Theorem 6.3.

Lemma 6.1. *The centralizer* $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ *of* PMod_S *in* Mod_S *is equal to the kernel of the action of* Mod_S *on* $V(S)$*. If* S *has positive genus, then* $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ *is equal to the kernel of the action of* Mod_S *on* $V_0(S)$*.*

Proof. Let $f \in \mathrm{Mod}_S$ and $\alpha \in V(S)$. Then $f t_\alpha f^{-1} = t_{f(\alpha)}$, where t_α is the (right) Dehn twist about α. On the other hand, $t_\beta = t_\alpha$ if and only if $\alpha = \beta$, by Theorem 4.1. Hence f commutes with t_α if and only if $f(\alpha) = \alpha$. Since PMod_S is generated by the Dehn twists about nontrivial circles, this proves the first assertion. If S has positive genus, then PMod_S is generated by the Dehn twists about nonseparating circles; cf. 2.1. This proves the second assertion. □

Lemma 6.2. *The centralizer* $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ *of* PMod_S *in* Mod_S *is a finite subgroup of* Mod_S*. It contains the centers* $C(\mathrm{Mod}_S)$ *and* $C(\mathrm{PMod}_S)$ *of* Mod_S *and* PMod_S *and is normal in* Mod_S*.*

Proof. Clearly, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ contains $C(\mathrm{Mod}_S)$ and $C(\mathrm{PMod}_S)$. Since PMod_S is obviously normal in Mod_S, the centralizer $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ is also normal. So, it remains to prove the first assertion.

If S is a sphere or a disc, then $\mathrm{PMod}_S = \mathrm{Mod}_S = \{1\}$. If S is an annulus, then $\mathrm{PMod}_S = 1$, $\mathrm{Mod}_S \cong \mathbb{Z}/2\mathbb{Z}$. Hence, if S is a sphere, a disc or an annulus, the first assertion is also clear.

If S is a torus or a torus with one hole, then $\mathrm{PMod}_S = \mathrm{Mod}_S \cong SL_2(\mathbb{Z})$. The center of $SL_2(\mathbb{Z})$ is well known to be equal to $\{-I, I\}$, where I is the 2×2 identity matrix. Hence, the first assertion is proved in this case also.

It remains to consider the case of a surface S of negative Euler characteristic.

Let C be a maximal system of circles on S, σ be the simplex of $C(S)$ corresponding to C and $R = S_C$. By Lemma 6.1, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) \subset M(\sigma)$. Hence, the reduction homomorphism r_C (cf. 2.3) gives rise to a homomorphism $r'_C : C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) \to \mathrm{Mod}_R$. The kernel of r'_C is equal to $T_C \cap C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$. Suppose that f is a nontrivial element of the kernel of r'_C. Such a element f is a nontrivial product of Dehn twists about certain components of C. Let $\alpha \in V(S)$ be the isotopy class of any one of these components of C. We may choose $\beta \in V(S)$ such that $i(\alpha, \beta) \neq 0$ and $i(\gamma, \beta) = 0$ for each vertex γ of σ other than α. Proposition 1 from [FLP], Expose [4], Appendice, implies that $i(f(\beta), \beta) \neq 0$. Hence, $f(\beta) \neq \beta$. Since $f \in C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$, this is impossible by Lemma 6.1. Hence, the kernel of r'_C is trivial and $r'_C : C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) \to \mathrm{Mod}_R$ is injective. Since each component of R is a disc with two holes, an element of Mod_R is determined by its action on the components of ∂R. Thus, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ is a finite group. This completes the proof. □

Theorem 6.3.

(i) *If* S *is an annulus, then* $C(\mathrm{PMod}_S) = \mathrm{PMod}_S = \{1\}$ *and* $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) = \mathrm{Mod}_S = \mathbb{Z}/2\mathbb{Z}$*.*

(ii)*If S is a disc with two holes, then* $C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) = \mathrm{PMod}_S = \{1\}$ *and* $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = \mathrm{Mod}_S$.

(iii) *If S is a torus with at most one hole or a closed surface of genus* 2, *then* $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) \cong \mathbb{Z}/2\mathbb{Z}$.

(iv) *If S is a sphere with four holes, then* $C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) = \{1\}$ *and* $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

(v)*If S is a torus with two holes, then* $C(\mathrm{PMod}_S) = \{1\}$ *and* $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) \cong \mathbb{Z}/2\mathbb{Z}$.

(vi) *Otherwise,* $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) = \{1\}$.

Proof. (i) is trivial.

(ii) If S is a disc with two holes then PMod_S is a trivial subgroup of Mod_S. Hence, $C(\mathrm{PMod}_S)$ is trivial and $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = \mathrm{Mod}_S$. In addition, Mod_S is isomorphic to the group of permutations of the three components of ∂S. It follows that $C(\mathrm{Mod}_S)$ is trivial. This proves (ii).

(iii) If S is a torus with at most one hole, then Mod_S is isomorphic to $SL_2(\mathbb{Z})$. The center of $SL_2(\mathbb{Z})$ is equal to $\{-I, I\}$, where I is the 2×2 identity matrix. Thus, $C(\mathrm{Mod}_S)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Since S has at most one boundary component, $\mathrm{PMod}_S = \mathrm{Mod}_S$. It follows that $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S)$. This proves the result for a torus with at most one hole.

If S is a closed surface of genus 2, then, since S has no boundary, $\mathrm{PMod}_S = \mathrm{Mod}_S$. Hence, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S)$. It is a standard fact that $C(\mathrm{Mod}_S)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$; cf. [M].

This proves the result for a closed surface of genus 2. For the proof of (iv) and (v), we need a detailed description of the generator of $C(\mathrm{Mod}_{S_2})$ for a closed surface S_2 of genus 2, which we will give now. This generator is known as the hyperelliptic involution of S_2. We will denote it by i. For some hyperbolic metric on S_2 there is an isometry $F : S_2 \to S_2$ of order 2 in the isotopy class i. (This isometry is, in fact, unique and exists for *any* hyperbolic metric on S_2, but we don't need these facts.) Let C be a system of three nonseparating circles on S_2, dividing S_2 into two discs with two holes, which we denote by P and Q. We may assume that the components of C are geodesic. By Lemma 6.1, $i(\alpha) = \alpha$ for all $\alpha \in V(S_2)$. Since there is a unique geodesic in each isotopy class $\alpha \in V(S_2)$, we conclude that F preserves each geodesic circle on S_2. In particular, F preserves three components a, b, c of C. Moreover, F interchanges P and Q (otherwise, i is a composition of Dehn twists about a, b and c and cannot be of order 2).

(iv) First, we will use the involution F of S_2 from the proof of (iii) in order to construct a nontrivial element of $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$. Let us cut S_2 along two circles a and b introduced above. We will get a surface S diffeomorphic to a sphere with four holes. Since F is an isometry preserving a and b, it induces an isometry $F_{a \cup b}$ of S. Any vertex $\delta \in V(S)$ is represented by a unique geodesic d on S. Such a geodesic d is simultaneously a geodesic on S_2 and, hence, $F(d) = d$. This implies that $F_{a\cup b}(d) = d$. Thus, the isotopy class $f_{a\cup b}$ of $F_{a\cup b}$ is in the kernel of the action of Mod_S on $V(S)$. By Lemma 6.1 this implies that $f_{a\cup b} \in C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$. The fact that F interchanges discs with two holes P and Q introduced above implies that $F_{a\cup b}$ induces a nontrivial permutation of components of ∂S. Thus, $f_{a\cup b}$ is a nontrivial element of the centralizer $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$.

Next, we prove that an element of $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ (where S is a sphere with four holes) is determined by its action on the components of ∂S. Clearly, it is sufficient to prove that the intersection $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) \cap \mathrm{PMod}_S$ is trivial. But, clearly, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) \cap \mathrm{PMod}_S = C(\mathrm{PMod}_S)$ and, hence, it is sufficient to prove that the center $C(\mathrm{PMod}_S)$ is trivial. Let $f \in C(\mathrm{PMod}_S)$. Let C be a nontrivial (separating) circle on S and let $R = S_C$. By Lemma 6.1, f preserves the isotopy class of C and, hence, we can speak about $f_C \in \mathrm{Mod}_R$. The surface R consisits of two components, which we will denote by P and Q. Each of them is a disc with two holes. Since f preserves all boundary components of S, the element f_C preserves both P and Q and, moreover, all boundary components of both P and Q (because two of three boundary components of, say, P are contained in ∂S and, hence, are preserved by f). Since both P and Q are discs with two holes, this implies that f_C is equal to the trivial element of Mod_R. In other words, f is in the kernel T_C of r_C. Since T_C is a free abelian group and f is of finite order by Lemma 6.2, we conclude that f is trivial. This proves that the center $C(\mathrm{PMod}_S)$ is, indeed, trivial.

Now, suppose that f is a nontrivial element of $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$. Then there exists a pair of distinct components a and b of ∂S such that $f(a) = b$. Let c and d be the remaining two components of ∂S. Choose a nontrivial circle E on S separating a and c from b and d. The circle E separates S into two components which we again denote by P and Q. Since $f(a) = b$, f must interchange P and Q. Hence, $f(c) = d$. Hence, the action of f on the components of ∂S is determined by its action on just one of them (say, a) and there are at most three nontrivial elements of $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$. These possible elements correspond to the permutations $(a, b)(c, d)$, $(a, c)(b, d)$ and $(a, d)(b, c)$ of the set $\{a, b, c, d\}$. Together with the trivial element, these permutations form a subgroup of the group of permutations Σ_4 of $\{a, b, c, d\}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Nontrivial elements of this subgroup form a complete conjugacy class in Σ_4. By the previous arguments, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ is nontrivial. Hence, at least one of the above permutations is realized. By conjugating with elements of Mod_S, we can realize the other two permutations. Hence, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

This completes our computation of $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ for sphere with four holes. We already saw that $C(\mathrm{PMod}_S) = \{1\}$. Since the center of the group of permutations of four components of ∂S (isomorphic to Σ_4) is trivial, $C(\mathrm{Mod}_S) \subset \mathrm{PMod}_S$. Thus, $C(\mathrm{Mod}_S) \subset C(\mathrm{PMod}_S)$ and $C(\mathrm{Mod}_S)$ is trivial. This completes the proof of (iv).

(v) The proof for the torus with two holes is similar to the proof for the sphere with four holes, but simpler. Again, we first construct a nontrivial element of $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$. This is done by cutting S_2 and F along just one of the circles a, b and c introduced in the proof of (iii). Next, we prove that an element of $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ (where S is a torus with two holes) is determined by its action on the components of ∂S. Again, it is sufficient to prove that the center $C(\mathrm{PMod}_S)$ is trivial. Let C be a system of two circles on S separating S into two discs with two holes P and Q. Any element of $C(\mathrm{PMod}_S)$ must preserve both components of ∂S and, by Lemma 6.1, both components of C. Hence, by the same argument as in the proof of (iv), $C(\mathrm{PMod}_S)$ is trivial. Now, an element of $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ is determined by its action on two components of ∂S and there is at least one nontrivial element of this centralizer. It follows that $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = \mathbb{Z}/2\mathbb{Z}$.

It remains to prove that $C(\mathrm{Mod}_S) = \mathbb{Z}/2\mathbb{Z}$. Let f be the nontrivial element of $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ (constructed above). Since f permutes two components of ∂S, any element $g \in \mathrm{Mod}_S$ can be written in the form $g = fg_1$, where $g_1 \in \mathrm{PMod}_S$. Since f commutes with itself and with g_1, it commutes also with g. Hence $C(\mathrm{Mod}_S)$ contains f and so is of order at least two. Finally, the fact that $C(\mathrm{Mod}_S) \subset C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ implies that $C(\mathrm{Mod}_S) = \mathbb{Z}/2\mathbb{Z}$. This completes the proof of (v).

(vi) Now we suppose that S is not a sphere with at most four holes, a torus with at most two holes or a closed surface of genus two. Let C be a maximal system of circles on S and let $R = S_C$, as in the proof of Lemma 6.2. Each component of R_C is a disc with two holes. Since S is not a torus with one hole, we may assume that each component of R is embedded in S. Let $f \in C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$. In view of Lemma 6.1 f preserves isotopy classes of all components of C. In particular, f_C is defined. Suppose there is a pair of distinct components a and b of ∂R such that $f_C(a) = b$. Let P be the component of R containing a.

Suppose first that a corresponds to a component c of C. Since f preserves the isotopy class of c, the component b also corresponds to c. Since P is embedded in S, at most one component of ∂P corresponds to c. Furthermore, there are exactly two components of R meeting along c. We denote them by P and Q. Hence, f_C must interchange P and Q. Since f preserves each component of C, surfaces P and Q must meet along those components of ∂P which correspond to components of C. All the other components of ∂P and ∂Q are components of ∂S. These considerations imply that S is the union of P and Q. (Note that $P \cup Q$ is a submanifold of S such that $\partial(P \cup Q) \subset \partial S$.) Since S is not a sphere with four holes, a torus with two holes or a closed surface of genus two, this is impossible. Hence, a cannot correspond to a component of C. Note that the same argument implies that f_C preserves each component of R.

Suppose now that a is a component of ∂S. Then $b = f_C(a)$ is also a component of ∂S. After changing, if necessary, C, we can assume that a and b lie in distinct components P and Q of R (since $a \neq b$). Clearly, $f_C(a) = b$ implies $f_C(P) = Q$. But, as we saw above, f_C preserves each component of R.

We see that f_C preserves all components of ∂R. This implies that $f_C = 1$ (because all components of R are discs with two holes). As before, Lemma 6.2 and the fact that T_C is free abelian imply that $f = 1$. Hence, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = \{1\}$. Since $C(\mathrm{Mod}_S), C(\mathrm{PMod}_S) \subset C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$, this completes the proof of (vi). □

7. A configuration of circles

In this Section we introduce a special configuration of circles, which will play an important role in the Sections 8 and 13. The most important property of this configuration is the fact that Dehn twists about the circles (of a subconfiguration) of this configuration generate the pure modular group PMod_S. Cf. Theorem 7.3.

7.1. The configuration $\mathcal{C}$. Let S be a compact orientable surface of positive genus which is not a (closed) torus. Let g be the genus of S and b the number of boundary components. We are interested in the configuration of cirles $\mathcal{C}$ presented on Figure 7.1. The configuration $\mathcal{C}$ is in minimal position and the intersection number $i(a, b)$ is 0 or 1 for each pair of circles a, b in $\mathcal{C}$. The circles $a_1, a_2, \ldots, a_{2g}$ form a chain in the sense that $i(a_i, a_{i+1}) = 1$ for $1 \leq i \leq 2g - 1$ and all other intersection numbers $i(a_j, a_k)$ are equal to 0. For any circle a_{2i} with $i \geq 2$ there are two circles b_{2i} and c_{2i} having the intersection number 1 with it and not belonging to the above chain. And for the last circle a_{2g} of the chain, there are additional $b - 1$ circles $d_1, d_2, \ldots, d_{b-1}$ having the intersection number 1 with it. All unmentioned intersection numbers are 0.

Note that if the genus $g = 1$, then there are no circles b_{2i} and c_{2i} for $1 \leq i \leq g$, if, $b = 1$, then there are no circles d_i, and if $b = 0$, there are no circles b_{2g} and c_{2g}.

The even-numbered circles $a_2, a_4, \ldots, a_{2g}$ of the above chain are called the *dual* circles of $\mathcal{C}$. They form a system of circles which we will denote by $\hat{C}$. If we remove $\hat{C}$ from $\mathcal{C}$, we obtain a collection of disjoint circles. We denote the corresponding system of circles by C. Clearly, C is a maximal system of circles.

Clearly, all components of S_C are discs with two holes. Moreover, all components of S_C are embedded in S, with only one exception of the case $g = b = 1$. For each component P of S_C the complement $S \setminus P$ is connected. Moreover, either ∂P consists of three components of of C or ∂P consisits of two components of C and one component of ∂S. In the first case ∂P is a system of circles on S. Recall that we call P an *interior* component in the first case and a *peripheral* component in the second (cf. 2.1). If there is an interior component, then the genus of S is at least 2.

Lemma 7.2. *Let S be a compact orientable surface of positive genus which is not a closed torus. Let S' be some other compact orientable surface. If S is a closed surface of genus 2, let us assume that S' is also a closed surface of genus 2.*

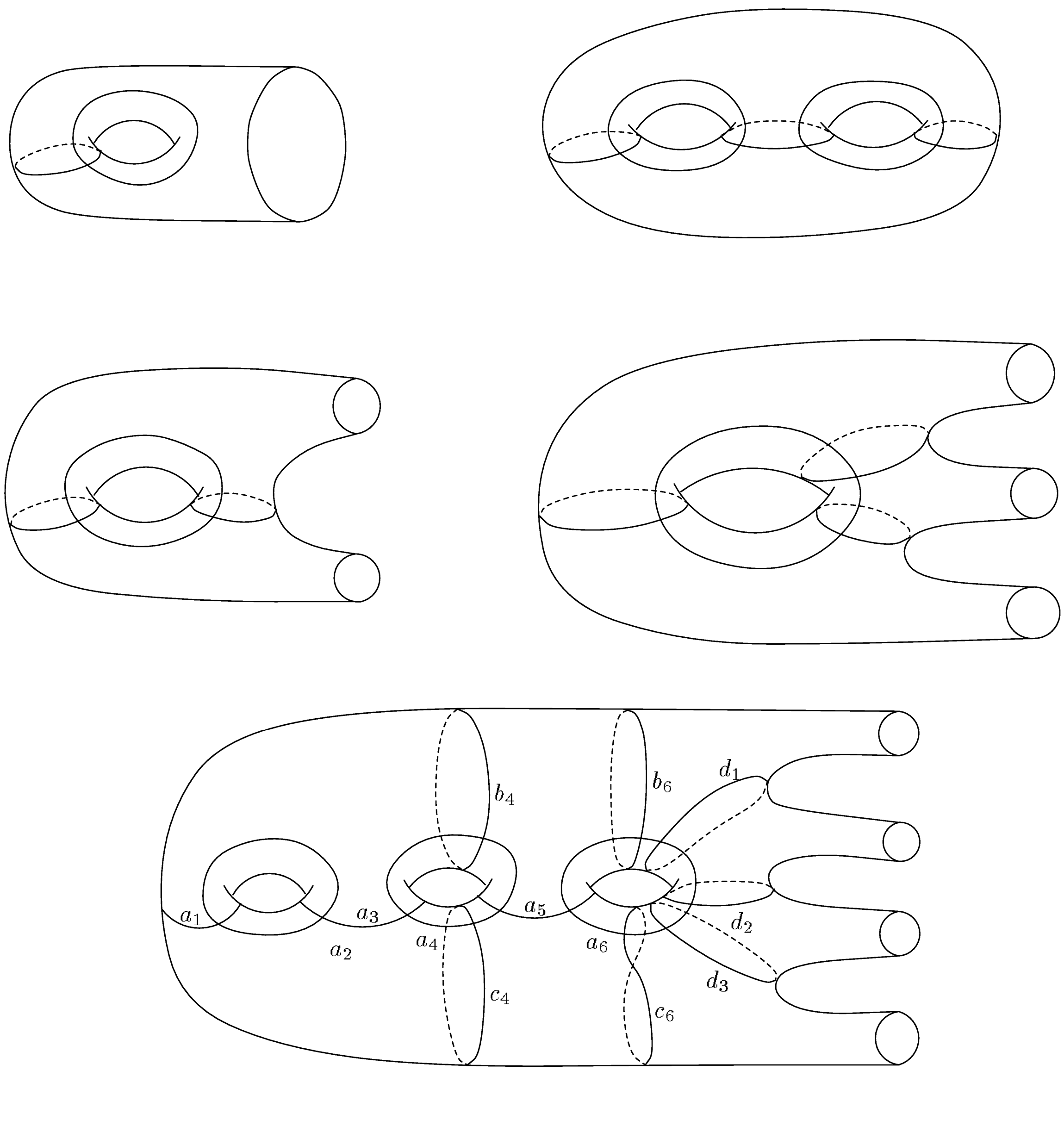


Figure 7.1

Let $x \mapsto x'$ be an injective map from the set of circles of the configuration $\mathcal{C}$ to the set of nontrivial circles on S'. Suppose that the configuration of circles $\mathcal{C}'$ formed by these circles x' is in minimal position and:

(i) *$i(x, y) = i(x', y')$ for all x, y in $\mathcal{C}$;*

(ii) *if three distinct circles x, y, z of $\mathcal{C}$ bound a disc with two holes in S, then the circles x', y', z' bound a disc with two holes in S'.*

Then there exist an embedding $H : S \to S'$ such that the image $H(S)$ contains all circles x'.

Suppose that, in addition, S' is diffeomorphic to S and:

(iii) *if x, y are adjacent cirles of the system of circles C, then x', y' are adjacent circles of the corresponding system of cirles C'.*

Then there exist a diffeomorphism $H : S \to S'$ such that $H(x) = x'$ for any circle x in $\mathcal{C}$.

Proof. Let us consider the case of a torus with one hole first. In this case there are only two circles, a_1 and a_2, in $\mathcal{C}$. Since $i(a'_1, a'_2) = i(a_1, a_2) = 1$ and the circles a'_1, a'_2 are in minimal position (i.e., in this case, are transversely intersecting at exactly one point), there is a neighborhood N' of $a'_1 \cup a'_2$ diffeomorphic to a torus with one hole, i.e. to S. Moreover, if S' is a torus with one hole also, we can choose N' to be equal to S' and find a diffeomorphism $H : S \to S'$ such that $H(a_1) = a'_1$, $H(a_2) = a'_2$. This proves the Lemma for a torus with one hole.

Next, let us consider the case of a torus with two holes. In this case there are only three circles, a_1, a_2, a_3, in $\mathcal{C}$. Since a'_1, a'_2, a'_3 are in minimal position, the assumption (i) implies the a'_2 intersects both a'_1 and a'_3 transversely in exactly one point and the circles a'_1 and a'_3 are disjoint. The existence of the required embedding or diffeomorphism in this situation is well known. Compare [I2], Lemma 5.1, or [M], Lemma 4.9.

In the remaining part of the proof we assume that S is not a torus with at most two holes. Note that in this case all components of S_C are embedded. We prove now a slightly strenghtened form of the first assertion. Namely, we prove that there exists an embedding $H : S \to S'$ such that:

(a) $H(x) = x'$ for circles x of the system of circles C with the exception of circles d_i, $1 \leq i \leq b-1$;

(b) $H(d_i) = d_{\sigma(i)}$ for all $i = 1, 2, \ldots, b-1$ and some permutation $\sigma : \{1, 2, \ldots, b-1\} \to \{1, 2, \ldots, b-1\}$;

(c) the image $H(S)$ contains the circles a'_{2i}, $1 \leq i \leq g$.

Again, there is a special case. Let S be a closed surface of genus 2. In this case the system of curves C consists of only three circles, a_1, a_3, a_5, and they divide S into two discs with two holes, which we denote by P and Q. By (ii), the circles a'_1, a'_3, a'_5 bound a disc with two holes P' in S'. Since S' is assumed to be a closed surface of genus 2 in this case, the complement $Q' = S' \setminus \operatorname{int} P'$ is also a disc with two holes.

Now, it is clear that there exists a diffeomorphism $H : S \to S'$ taking P to P', Q to Q' and a_i to a'_i for $i = 1, 3, 5$. Since there are no circles b_i, c_j or d_k in this case, this proves our assertion (including the first assertion of the Lemma).

Suppose now that S is not a closed surface of genus 2 (and not a torus with at most two holes). If P is an interior component of S_C and x, y, z are the circles of C corresponding to the components of ∂P, then, by (ii), x', y', z' bound a disc with two holes in S'. Let us prove that they bound only one such disc. Otherwise, S' is equal to the union of two discs with two holes bounded by x', y', z' and, hence, is a closed surface of genus 2. Since a system of circles on a closed surface of genus 2 contains no more than three circles, this implies that C' contains only x', y', z' and, hence, C contains only x, y, z (by the injectivity assumption). But, if S is not a closed surface of genus 2 and there is an interior component of S_C, the system of circles C contains at least four circles, namely a_1, a_3, b_4, c_4. The contradiction shows that x', y', z' bound only one disc with two holes, which we will denote by P' (if P is the disc with two holes bounded by x, y, z). The correspondence $P \mapsto P'$ is 1–1, because different interior components have different sets of boundary circles.

Let us orient S and S'. For any interior component P, let us choose an orientation-preserving diffeomorphism $P \to P'$ respecting the correspondence $x \mapsto x'$ (i.e. such that if a component of ∂P corresponds to x, then its image corresponds to x'). For any circle x from C different from $b_{2g}, c_{2g}, d_1, d_2, \dots, d_{b-1}$, these diffeomorphisms induce two diffeomorphisms $x \to x'$, coming from two sides of x. In fact, these two diffeomorphisms are always isotopic. This is because diffeomorphisms $P \to P'$ are orientation-preserving and two diffeomorphisms between two (oriented) circles are isotopic if and only if they are both orientation-preserving or both orientation-reversing. It follows that, after changing some of the diffeomorphisms $P \to P'$ by an isotopy, we may assume that diffeomorphisms $x \to x'$ coming from two sides of x are always equal. Then we can glue these diffeomorphism $P \to P'$ into a diffeomorphism $H_0 : S_0 \to S'_0$, where S_0 (respectively, S'_0) is the result of glueing of all interior components P (respectively, of the corresponding components P'). If ∂S is nonempty, then S_0 is a surface of genus $g - 1$ bounded by b_{1g} and c_{2g}, and S'_0 is a surface of genus $g-1$ bounded by b'_{1g} and c'_{2g}. If S is closed, then $S = S_0$ and $S' = S'_0$. In particular, if S is closed, we can take the diffeomorphism H_0 as the embedding H we are looking for now.

The next step is to extend the diffeomorphism $S_0 \to S'_0$ to the (image in S of the) peripheral components. Note that the intersection $S_0 \cap a_{2g}$ is an interval in a_{2g}. Let I_{2g} be the complementary closed interval in a_{2g}. The circles $d_1, d_2, \dots, d_{b-1}$ intersect a_{2g} in I_{2g}. Let us consider what happens with the corresponding circles on S'. Since a'_{2g} intersects both b'_{1g} and c'_{2g} in exacly one point, the intersection $S' \cap a'_{2g}$ is an interval in a'_{2g}. Let I'_{2g} be the complementary closed interval in a'_{2g}. Note that the circles $d'_1, d'_2, \dots, d'_{b-1}$ cannot intersect a'_{2g} in $S' \cap a'_{2g}$. In fact, if d'_i intersects a'_{2g} in $S' \cap a'_{2g}$, then d'_i is contained in S'_0, disjoint from $S'_0 \cap C$ and is not isotopic to any component of $S'_0 \cap C$. But this is impossible because, by the construction, $S'_0 \cap C$ divides S'_0 into several discs with two holes. Hence, the circles $d'_1, d'_2, \ldots, d'_{b-1}$ intersect a'_{2g} in I'_{2g}, each exactly at one point. But they do not necessary follow in the order $d'_1, d'_2, \ldots, d'_{b-1}$ along I'_{2g}. The new order gives rise to a permutation $\sigma : \{1, 2, \ldots, b-1\} \to \{1, 2, \ldots, b-1\}$ such that the circles follow in the order $b'_{2g}, d'_{\sigma^{-1}(1)}, d'_{\sigma^{-1}(2)}, \ldots, d'_{\sigma^{-1}(b-1)}, c'_{2g}$ along I'_{2g}. Now, by a standard argument, we can find a diffeomorphism $H_1 : N \to N'$ between a regular neighborhood N in S of the union $S_0 \cup I_{2g} \cup d_1 \cup d_2 \cup \ldots \cup d_{b-1}$ and a regular neighborhood N' in S' of the union $S'_0 \cup I'_{2g} \cup d'_1 \cup d'_2 \cup \ldots \cup d'_{b-1}$. Compare [I2], Lemma 5.1, or [M], Lemma 4.9 again. Moreover, we can choose H_1 to be equal to H_0 on S_0, mapping I_{2g} to I'_{2g} and d_i to $d_{\sigma(i)}$ for $1 \leq i \leq b-1$. And, since S is a regular neighborhood of the union $S_0 \cup I_{2g} \cup d_1 \cup d_2 \cup \ldots \cup d_{b-1}$, we can take S as N. Then, composing H_1 with the inclusion $N' \to S'$, we will get an embedding H. Clearly, H has the properties (a) and (b). Also, it follows from (i) that the circles a'_{2i}, $1 \leq i \leq g-1$ are contained in S'_0. Since $a'_{2g} \subset N'$, the property (c) follows. This completes the proof of our strenghtened form of the first assertion.

Let us prove now the second assertion of the Lemma. We prove first that (if S' is diffeomorphic to S, then) H has the following property:

(a-b) $H(x) = x'$ for all circles x of the system of circles C.

Note that since both circles b'_{2g} and c'_{2g} are nonseparating (they intersect a'_{2g} transversely in exactly one point), the union $b'_{2g} \cup c'_{2g}$ divides S' into eaxctly two parts, one of which is, obviously, S'_0. Let $S'_1 = S' \setminus \operatorname{int} S'_0$ be the other part. It follows from the classification of surfaces that S'_1 is a sphere with $b+2$ holes. Because the maximal number of circles in a system of circles on a sphere with $b+2$ holes is $b-1$, the circles $d'_1, d'_2, \ldots, d'_{b-1}$ form a maximal system of circles on S'_1. In particular, $d'_1, d'_2, \ldots, d'_{b-1}$ divide S'_1 into b discs with two holes, which we denote by $P'_1, P'_2, \ldots, P'_b$.

For each P'_i, no more than one component of the boundary corresponds to a component of $\partial S'$. In fact, if all three components of $\partial P'_i$ correspond to components of of ∂S, then P'_i has to be a connected component of S, and if exactly two components of $\partial P'_i$ correspond to components of ∂S, then the third component corrresponds to a separating circle on S, contradicting the fact all circles $b'_{2g}, d'_1, d'_2, \ldots, d'_{b-1}, c'_{2g}$ are nonsepaprating. Since there are b boundary components of S' and b surfaces P'_i, we see that exactly one boundary component of each P'_i corresponds to a component of $\partial S'$.

Consider two consequtive circles x, y from the sequence $b_{2g}, d_1, d_2, \ldots, d_{b-1}, c_{2g}$. Clearly, x and y are adjacent circles of the system of circles C. By (iii), x' and y' are adjacent circles of C'. Since at least one of the circles x', y' is not contained in S'_0, this means that for some P'_i both x' and y' correspond to boundary components of P'_i. The third boundary component of P'_i corresponds to some boundary component of S'. Now, let us look at the intersection of a'_{2g} with the image of P'_i in S'. It is a one-dimensional manifold with boundary and without closed components. The boundary is contained in $x' \cup y'$ and, hence, consists of two points. It follows that this intersection is an arc connecting x' with y'. Hence, y' follows x' along a'_{2g}. Since this is true for any consequtive x, y, we see that the above permutation σ is, in fact, trivial in the our case. In other words, H has the property (a-b).

Notice that each peripheral componet P_j of S_C is embedded by H into some P'_i. Since this embedding induces a diffeomorphism on two boundary components of P_j (the ones coming from C), we can modify H in such a way that all these embeddings will be diffeomorphisms. Then the embedding H itself will be a diffeomorphism.

It remains to consider the dual circles $a_2, a_4, \ldots, a_{2g}$. If x is one of them, then $H(x)$ and x' intersect the same circles of C' and all these intersections are transverse and one-point. Since $S'_{C'}$ is a union of discs with two holes, the well known Dehn–Thurston classification of multi-circles implies that, up to an isotopy preserving C', the collections of circles $\{H(x) : x \text{ is a dual circle }\}$ and $\{x' : x \text{ is a dual circle }\}$ differ by a composition of twist maps about components of C'. Hence, by composing H with such a composition and then applying some isotopy, we will get a new diffeomorphism H, which will satisfy (iii). This proves the second assertion of the Lemma. □

Theorem 7.3. PMod_S *is generated by the Dehn twists along the circles of the configuration* $\mathcal{C}$.

Proof. We use the induction by the number of holes. If there is no holes (i.e., $b = 0$), then our set of Dehn twists contains the well known set of Lickorish generators and, hence, the Theorem is true in this case.

Suppose now that $b > 0$. Let R be the surface obtained from S by glueing a disc D to some boundary component of S, say, to the one situated between b_{2g} and d_1. Extending diffeomorphisms from S to R defines a canonical map $E : \mathrm{PMod}_S \to \mathrm{PMod}_R$. If S is a torus with one hole, this map is an isomorphism and the Theorem follows. Otherwise (under our assumption that S is of positive genus, cf. 7.1), there is a natural short exact sequence

$$1 \to \pi_1(R) \xrightarrow{\partial} \mathrm{PMod}_S \xrightarrow{E} \mathrm{PMod}_R \to 1.$$

For a discussion of this exact sequence, as soon as for the proof of the following two facts about the homomorphism ∂ from it, see, for example, [I2], Section 6.1. In discussing these properties of ∂, it is convenient to assume that the base point x for the fundamental group $\pi_1(R) = \pi_1(R, x)$ is contained in the interior of the attached disc D and that the diffeomorphisms extended from S to R fix this base point x. Then the extended diffeomorphisms act on $\pi_1(R) = \pi_1(R, x)$. This action gives rise to a well defined action of PMod_S on $\pi_1(R)$, which we will denote by a dot.

First we need a description of the image $\partial([l])$, where $[l]$ is the homotopy class of an embedded loop l in R based at x. We consider (without any real loss of generality) only the case when the image of l intersects D in an arc meeting the boundary of D

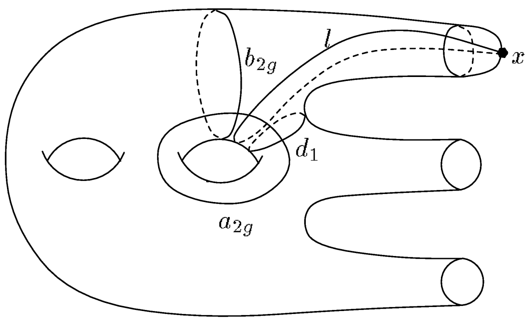


Figure 7.2

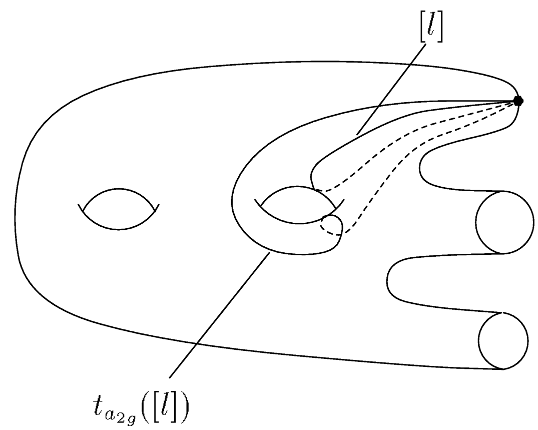
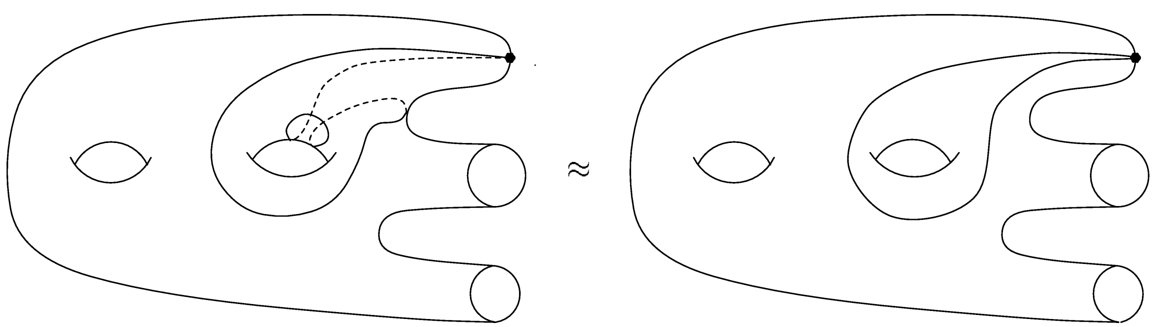
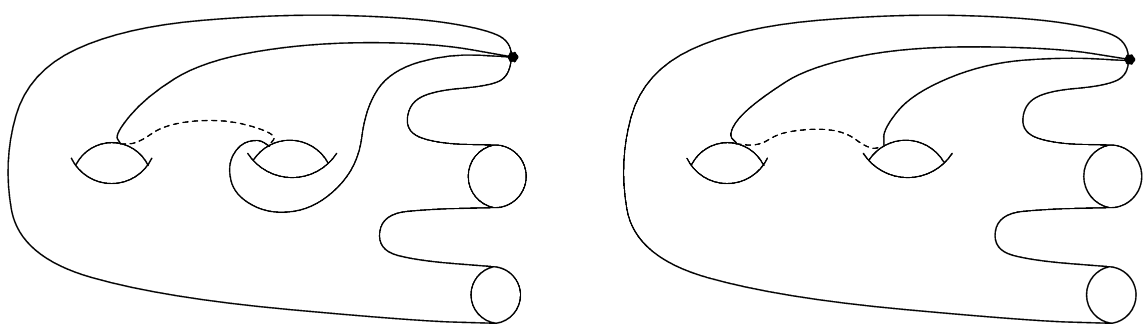


Figure 7.3



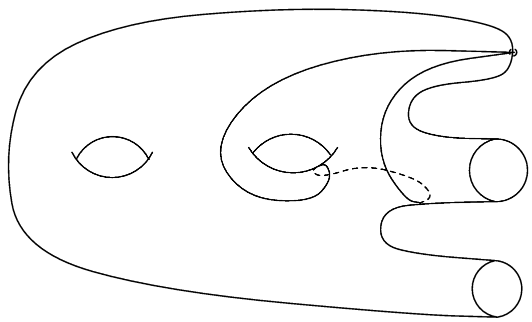
$t_{d_1} \circ t_{a_{2g}}([l])$

Figure 7.4



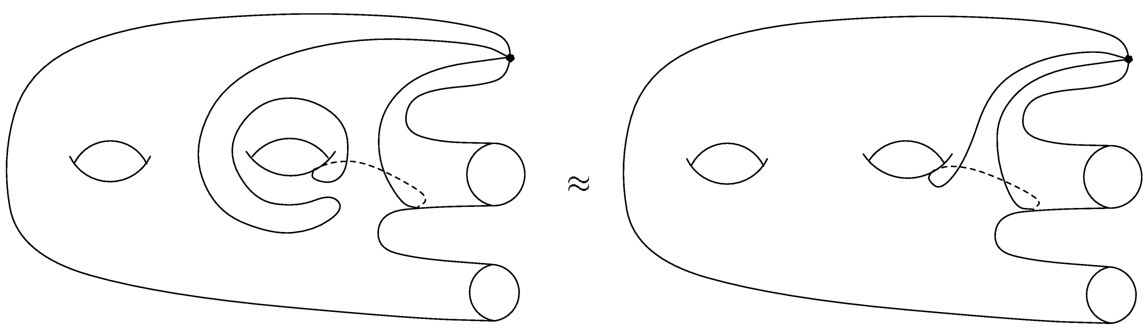
$t_{a_{2g-1}} \circ t_{d_1} \circ t_{a_{2g}}([l])$ $(t_{a_{2g-1}} \circ t_{d_1} \circ t_{a_{2g}} \circ [l])(t_{d_1} \circ t_{a_{2g}}([l]))^{-1}$

Figure 7.5



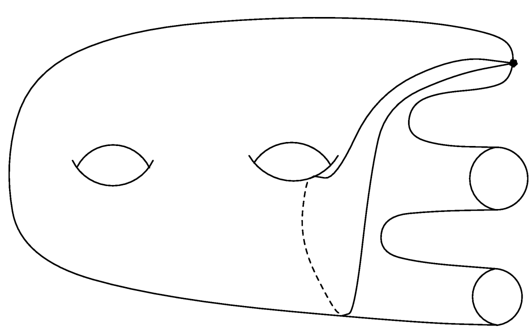
$t_{d_2}^{-1} \circ t_{d_1} \circ t_{a_{2g}}([l])$

Figure 7.6



$t_{a_{2g}}^{-1} \circ t_{d_2}^{-1} \circ t_{d_1} \circ t_{a_{2g}}([l])$

Figure 7.7



$t_{a_{2g}}^{-1} \circ t_{d_3}^{-1} \circ t_{d_1} \circ t_{a_{2g}}([l])$

Figure 7.8

only at its two endpoints. Then we can find an annulus A in R containing D in its interior and having the image of l as its axis. Let a be the boundary component of A situated to the right of l with respect to some fixed orientation of S (notice that l is a map $[0,1] \to R$ and its image is naturally oriented), and let b the boundary component situated to the left of l. Then

$$\partial([l]) = t_a t_b^{-1}.$$

The second fact we need is the following formula relating ∂ and the above action of PMod_S on $\pi_1(R)$:

$$\partial(f \cdot a) = f \partial(a) f^{-1}.$$

The extension homomorphism $E : \mathrm{PMod}_S \to \mathrm{PMod}_R$ maps our Dehn twists along the circles in $\mathcal{C}$ into the corresponding Dehn twists in PMod_R (note that both $t_{b_{2g}}$ and t_{d_1} are mapped to $t_{b_{2g}}$) which generate PMod_R by the inductive assumption (since R has one hole less than S). Hence, it is sufficient to show that the group G generated by the Dehn twists along the circles of the configuration $\mathcal{C}$ contains $\mathrm{Ker}\, E = \mathrm{Im}\, \partial$. It is sufficient to show that G contains a set of generators of this image $\mathrm{Im}\, \partial$. To begin with, G contains $t_{b_{2g}} t_{d_1}^{-1} = \partial([l])$, where l is the loop represented on Figure 7.2. In view of the above relation between ∂ and the action of PMod_S on $\pi_1(R)$, it is sufficient to show that the group generated by the PMod_S-orbit of $[l]$ contains a set of generators of $\pi_1(R)$. Now, several elements of this group are calculated on Figures 7.3–7.8. Among them are some of the standard generators of $\pi_1(R)$, namely, the ones on Figures 7.4, 7.7, 7.8. The loop on Figure 7.2 also represents a standard generator. Clearly, by continuing these calculations in the same way, we will eventually get a set of generators for $\pi_1(R)$. It follows that G contains a set of generators of $\partial(\pi_1(R))$. This implies that our Dehn twists generate PMod_S. The induction completes the proof. □

7.4. Remark. The Lickorish generators for the case $b = 0$ include only a part of our generators. In fact, the circles c_{2i} are not needed. By following the above proof it is easy to see that these circles are not need in the case $b > 0$ either.

8. Twist-preserving homomorphisms

In this section, S and S' denote compact connected *oriented* surfaces. We assume that S has positive genus and is not a closed torus.

Let $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ or $\mathrm{PMod}_S \to \mathrm{Mod}_{S'}$ be an injective homomorphism. We say that ρ is *twist-preserving* if $\rho(t_\alpha)$ is a right Dehn twist about a nontrivial circle on S' for each $\alpha \in V_0(S)$. In other words, ρ is twist-preserving if for each $\alpha \in V_0(S)$, there exists an isotopy class $\rho(\alpha) \in V(S')$ such that $\rho(t_\alpha) = t_{\rho(\alpha)}$. By Theorem 4.1, $\rho(\alpha)$ is uniquely determined by the identity $\rho(t_\alpha) = t_{\rho(\alpha)}$.

This section is devoted to injective twist-preserving homomorphisms $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ or $\mathrm{PMod}_S \to \mathrm{Mod}_{S'}$. We would like to prove that such a homomorphism is,

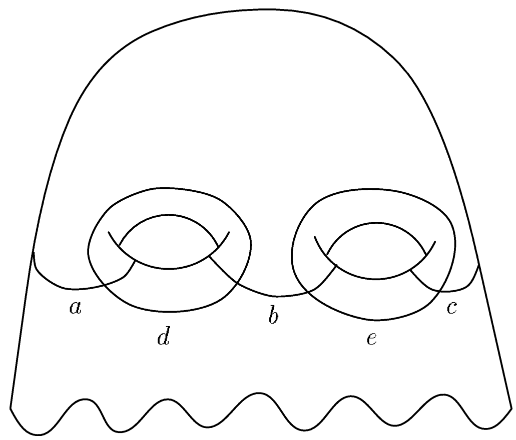


Figure 8.1

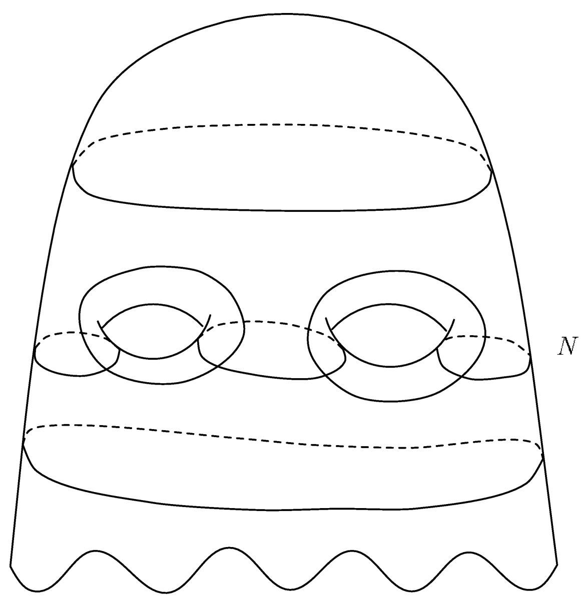


Figure 8.2

in fact, induced by a diffeomorphism $S \to S'$. We will actually do this under two different (but overlapping) assumptions: if the genus of S is at least two, and if the genus of S is at least one and the maxima of ranks of maximal abelian subgroups of S and S' differ by at most one. The first case is dealt with in Theorem 8.9, and the second one in Theorem 8.15. These two theorems are the main results of this section. Both of them require a lot of preliminary work, done in Lemmas and Corollaries 8.1–8.8 and in Lemmas 8.10–8.14 respectively.

For the remainder of this section, we assume that ρ is twist-preserving homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ or $\mathrm{PMod}_S \to \mathrm{Mod}_{S'}$. Let a be a nonseparating circle on S. For a nonseparating circle a on S, we will denote by $\rho(a)$ some representative of the isotopy class $\rho(\alpha)$, where α is the isotopy class of a. Then $\rho(a)$ is well defined up to isotopy on S' and $\rho(t_a) = t_{\rho(a)}$.

Lemma 8.1. *$\rho(\alpha) = \rho(\beta)$ if and only if $\alpha = \beta$.*

Proof. The "if" clause is trivial. Suppose that $\rho(\alpha) = \rho(\beta)$. This means that $\rho(t_\alpha) = \rho(t_\beta)$. Since ρ is injective, this implies that $t_\alpha = t_\beta$. Hence, by Theorem 4.1, $\alpha = \beta$. This completes the proof. □

Lemma 8.2. *Let a and b be distinct nonseparating circles on S. Then:*
(i) *$i(\rho(a), \rho(b)) = 0$ if and only if $i(a, b) = 0$;*
(ii) *$i(\rho(a), \rho(b)) = 1$ if and only if $i(a, b) = 1$.*

Proof. It follows from trom Theorem 4.2 that $i(a, b) = 0$ if and only if t_a and t_b commute and $i(a, b) = 1$ if and only if $t_a t_b t_a = t_b t_a t_b$. The result follows from the fact that ρ is an injective homomorphism. □

Corollary 8.3. *$\rho(a)$ is nonseparating for every nonseparating circle a on S.*

Corollary 8.4. *Let C be a system of nonseparating circles on S and σ be the corresponding simplex of $C(S)$. Let $\rho(\sigma) = \{\rho(\alpha) : \alpha \in \sigma\}$. Then $\rho(\sigma)$ is a simplex of $C(S')$.*

Lemma 8.5. *Let P be a disc with two holes embedded in S and such that $S \setminus P$ is connected and ∂P is a system of circles on S. Let a, b and c be the three boundary components of P. In view of Corollary 8.4 $\{\rho(a), \rho(b), \rho(c)\}$ is a simplex and, hence, we may assume that circles $\rho(a)$, $\rho(b)$ and $\rho(c)$ are disjoint. Then $\rho(a)$, $\rho(b)$ and $\rho(c)$ bound a disc with two holes P' embedded in S' such that $S' \setminus P'$ is connected and $\partial P'$ is a system of circles on S'.*

If, in addition, S is a closed surface of genus 2, then S' is also a closed surface of genus 2 and, hence, S' is a union of P' and another disc with two holes Q' meeting P' along their common boundary.

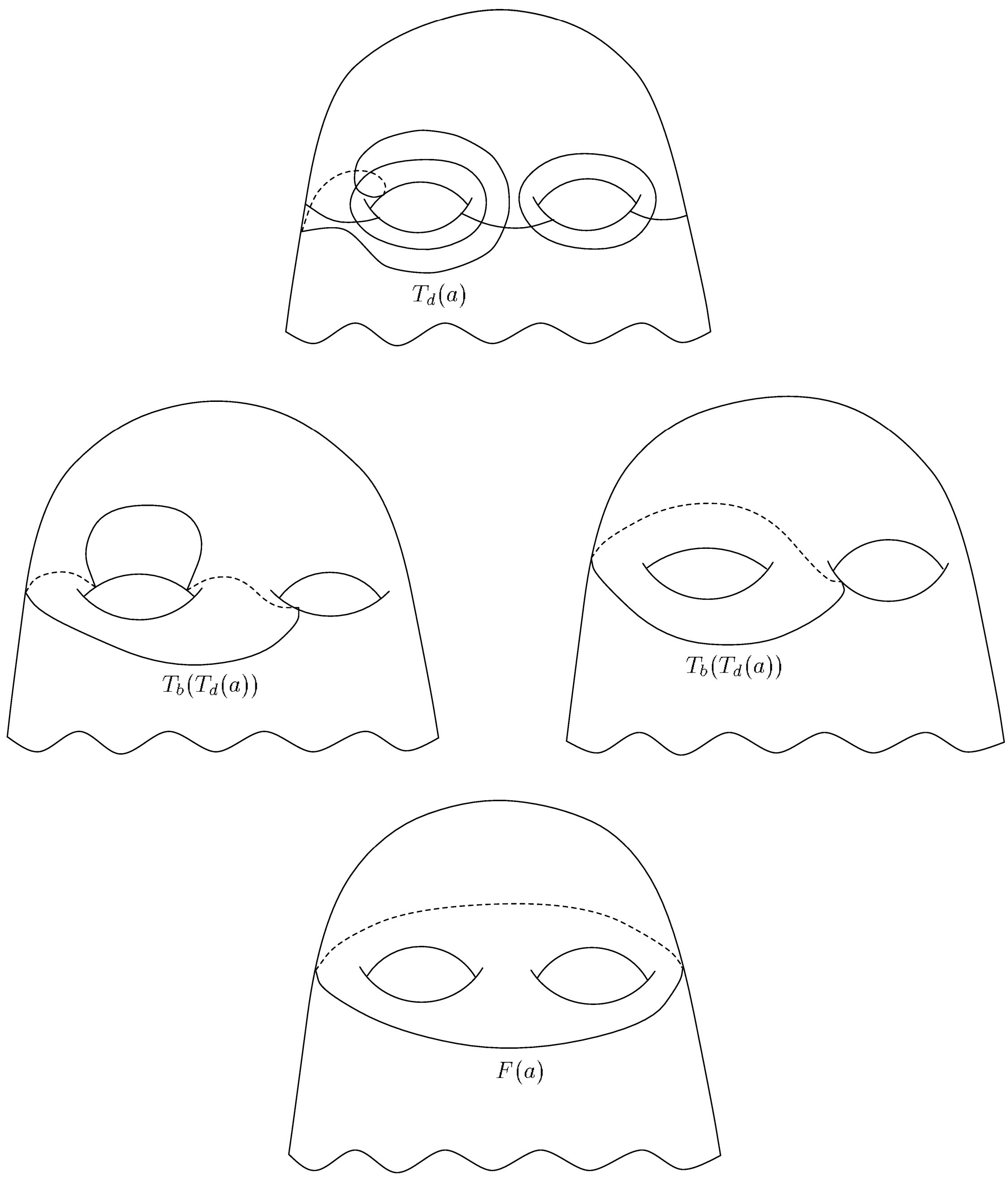


Figure 8.3

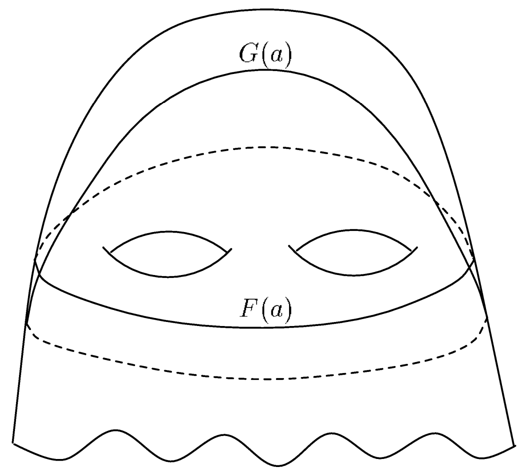


Figure 8.4

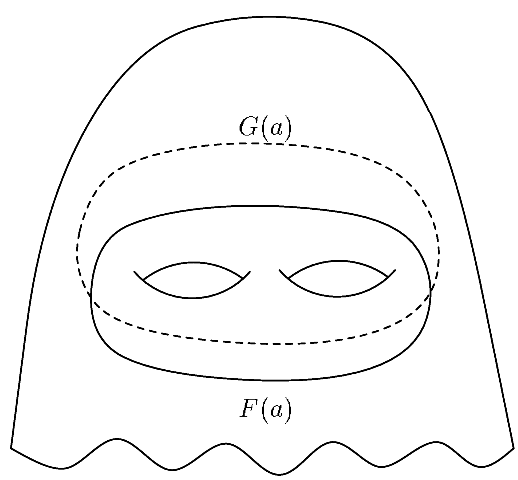


Figure 8.5

Proof. Since S and $S \setminus P$ are connected, the topological type of $S \setminus \operatorname{int} P$ is determined by the topological type of S (and the fact that P is a disc with two holes). It follows that up to a diffeomorphism the pair (S, P) is determined by S. By looking at one such pair on Figure 8.1, we see that we may choose a pair of nontrivial circles d and e on S such that

$$i(a, d) = i(d, b) = i(b, e) = i(e, c) = 1$$

and

$$i(d, e) = i(d, c) = i(e, a) = 0.$$

We assume that the circles a, b, c, d and e are in minimal position. It is clear that all circles a, b, c, d and e are nonsepaprating (since any one of them intersects some other transversely at one point).

By Lemma 8.2, the circles $\rho(a)$, $\rho(b)$, $\rho(c)$, $\rho(d)$ and $\rho(e)$ have the same pairwise geometric intersection numbers as a, b, c, d and e. Clearly, we may assume that $\rho(a)$, $\rho(b)$, $\rho(c)$, $\rho(d)$ and $\rho(e)$ are in minimal position.

Let N (respectively N') be a neighborhood of the union $a \cup b \cup c \cup d \cup e$ (respectively $\rho(a) \cup \rho(b) \cup \rho(c) \cup \rho(d) \cup \rho(e)$) diffeomorphic to a torus with two holes and containing this union as a deformation retract. Cf. Figure 8.2.

As is well known, our assumtions on the intersection numbers and the fact that our circles are in minimal position imply that there exist a diffeomorphism $H : N \to N'$ such that $H(x) = \rho(x)$ for each $x = a$, b, c, d or e. Compare [I2], Lemma 5.1, or [M], Lemma 4.9. (In both [I2] and [M] only *maximal chains* are considered, but the proofs work with trivial changes for all *chains*. The sequence of circles a, d, b, e, c is a particular example of a chain; the reader can figure out the general definition without any trouble.)

If $F : S \to S$ is a diffeomorphism with support in N, then we will denote by F^H the diffeomorphism $S' \to S'$ equal to $H \circ F|N \circ H^{-1}$ on N' and equal to the identity outside N'. For example, if T_x is a twist map about a circle in N (always assumed to have support in N), then T_x^H is a twist map about the circle $H(x)$. In particular, if $x = a$, b, c, d or e, then T_x^H is a twist map about $\rho(a)$, $\rho(b)$, $\rho(c)$, $\rho(d)$ or $\rho(e)$ respectively. We denote it also by $T_{H(x)}$.

Let $F = T_c \circ T_e \circ T_b \circ T_d$ and $G = T_c^{-1} \circ T_e^{-1} \circ T_b^{-1} \circ T_d^{-1}$, and let us consider the isotopy classes of circles $F(a)$ and $G(a)$ in N and in S. A circle isotopic to $F(a)$ in N is found on Figure 8.3. This circle together with a circle isotopic to $G(a)$ is shown on Figure 8.4. It is clear that the circles on Figure 8.4 are in minimal position in N and, hence, have the intersection number 2 there. On the other hand, these circles are isotopic on S to the disjoint circles shown on Figure 8.5, and, hence, have the intersection number 0 on S. These properties are crucial for our proof.

Let f and g be the isotopy classes of F and G respectively. Clearly, $f = t_c t_e t_b t_d$ and $g = t_c^{-1} t_e^{-1} t_b^{-1} t_d^{-1}$. Consider now $\varphi = f t_a f^{-1}$ and $\psi = g t_a g^{-1}$. These two elements are Dehn twists about $F(a)$ and $G(a)$, respectively. On the other hand, φ and ψ are products of several Dehn twists about circles a, b, c, d and e. This implies that $\rho(\varphi)$ and $\rho(\psi)$ are similar products of Dehn twists about circles $\rho(a)$, $\rho(b)$, $\rho(c)$, $\rho(d)$ and $\rho(e)$. This implies, in turn, that we can take $(F \circ T_a \circ F^{-1})^H$ and $(G \circ T_a \circ G^{-1})^H$ as representatives of $\rho(\varphi)$ and $\rho(\psi)$ respectively. Since $F \circ T_a \circ F^{-1}$ (respectively $G \circ T_a \circ G^{-1}$) is a twist map about $F(a)$ (respectively $G(a)$, it follows that $\rho(\varphi)$ (respectively $\rho(\psi)$) is a Dehn twist about $H(F(a))$ (respectively $H(G(a))$.

Clearly, the circles $H(F(a))$ and $H(G(a))$ have the same intersection number in N' as the circles $F(a)$ and $G(a)$ have in N. If none of the two boundary components of N' bounds a disc in S', these circles have the same intersection number in S' also. Hence, in this case the intersection number of $H(F(a))$ and $H(G(a))$ is 2 and Dehn twists $\rho(\varphi)$ and $\rho(\psi)$ about these circles do not commute (cf. Theorem 4.2). On the other hand, the elements φ and ψ are Dehn twist about circles $F(a)$ and $G(a)$, and, since these circles have the intersection number 0 in S, these two elements φ and ψ commute in Mod_S. Since ρ is a homomorphism, this implies that $\rho(\varphi)$ and $\rho(\psi)$ commute. The contradiction we reached means that at least one of the boundary components of N' bounds a disc in S'. Clearly, this implies the first assertion of the Lemma.

Let us prove the second assertion. So, we assume now that S is a closed surface of genus 2. Let U (respectively V) be a neighborhood of $a \cup d$ (respectively $c \cup e$) in S diffeomorphic to a torus with one hole and containing this union as a deformation retract. We may assume that U and V are disjoint. Let u, v be the boundary circles of U, V respectively. By Theorem 4.3 $(t_a t_d)^6 = t_u$ and $(t_e t_c)^6 = t_v$. On the other hand, u is isotopic to v on S because S is a closed surface of genus 2; cf. Figure 8.6. It follows that $t_u = t_v$ and $(t_a t_d)^6 = (t_e t_c)^6$. Hence $(\rho(t_a)\rho(t_d))^6 = (\rho(t_e)\rho(t_c))^6$.

Note that $(\rho(t_a)\rho(t_d))^6$ is represented by $(T_a^H \circ T_d^H)^6 = (T_{H(a)} \circ T_{H(d)})^6$. Applying Theorem 4.3 to S', we see that $(T_{H(a)} \circ T_{H(d)})^6$ is isotopic to a twist map $T_{H(u)}$ about $H(u)$. Hence, $(\rho(t_a)\rho(t_d))^6 = t_{H(u)}$. Similarly, $(\rho(t_e)\rho(t_c))^6 = t_{H(v)}$. Since $(\rho(t_a)\rho(t_d))^6 = (\rho(t_e)\rho(t_c))^6$, we conclude that $t_{H(u)} = t_{H(v)}$. By Theorem 4.1, $H(u)$ is isotopic to $H(v)$. Since $H(u)$ and $H(v)$ are disjoint (because U and V are), they bound an annulus. The union of $H(U)$, $H(V)$ and this annulus is a closed surface of genus 2 contained in S'. Clearly, it has to be equal to the surface S' itself. Now, the fact that S' is closed surface of genus 2 and that P' is a disc with two holes implies that $Q' = S' \setminus \operatorname{int} P'$ is also a disc with two holes. This completes the proof of the second assertion of the lemma. □

Lemma 8.6. *Let C be a system of nonseparating circles on S and σ be the corresponding simplex of $C(S)$. Let $\rho(\sigma)$ be the corresponding simplex of $C(S')$ as in Corollary 8.4 and let $\rho(C)$ be a realization of $\rho(\sigma)$. If a and b are adjacent components of C, then $\rho(a)$ and $\rho(b)$ are adjacent components of $\rho(C)$.*

Proof. Recall (cf. 2.1) that a and b are adjacent if there exists a component Q of S_C such that a and b both correspond to components of ∂Q. In this case there

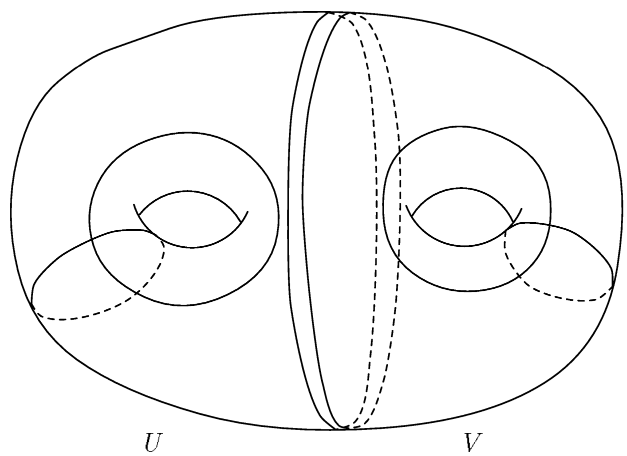


Figure 8.6

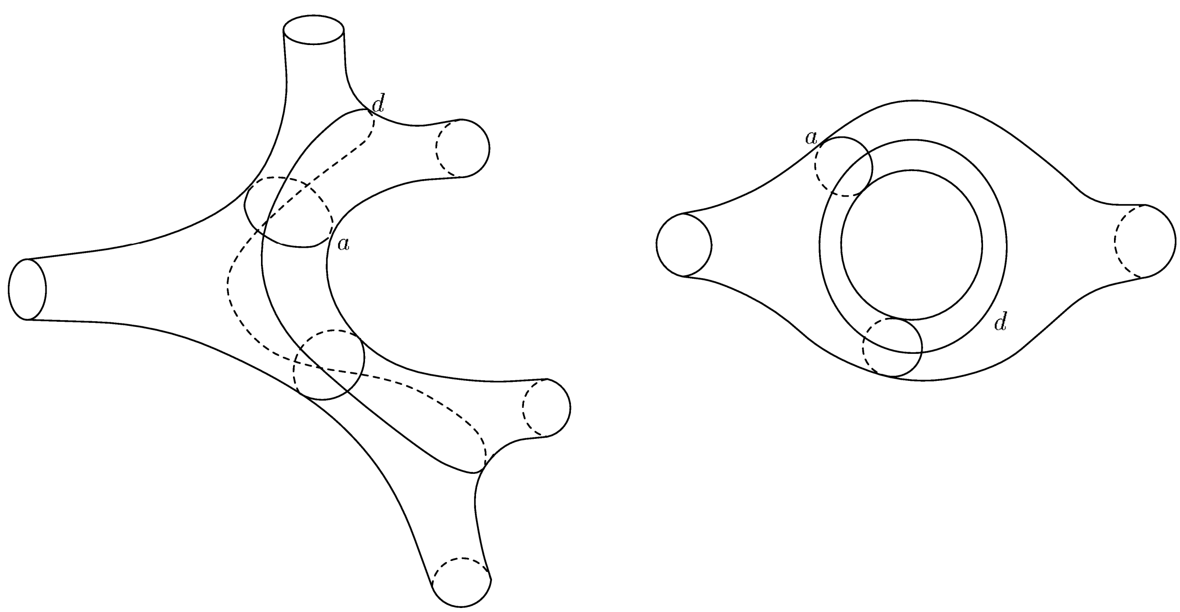


Figure 8.7

exists a nontrivial circle d on S such that $i(d,a), i(d,b) \neq 0$ and $i(d,c) = 0$ for every component $c \neq a, b$; cf. Figure 8.7.

Let us show that we always can replace d by a nonseparating circle with the same properties. Suppose that d is separating. Let d' be the image of a under a twist map about d. Clearly, $i(d', c) = 0$ for every component c of C different from a, b. Since d' is the image of a under a diffeomorphism of S and a is nonseparating, d' is nonseparating. By a special case of Proposition 1 from [FLP], Expose [4], Appendice, $i(d', a) = i(d, a)^2$ and $i(d', b) = i(d, a)i(d, b)$. Hence, d' is the desired circle.

So, we may assume that d is nonseparating. Then, $\rho(d)$ is defined. By Lemma 8.2, $i(\rho(d), \rho(a)), i(\rho(d), \rho(b)) \neq 0$ and $i(\rho(d), \rho(c)) = 0$ for every component c of C different from a, b. Hence, $\rho(d)$ is isotopic to a circle whose intersection with each of $\rho(a)$ and $\rho(b)$ is nonempty but whose intersection with $\rho(c)$ is empty for each component c of C different a, b. The existence of such a circle implies that $\rho(a)$ and $\rho(b)$ are adjacent components of C'. This completes the proof. □

Lemma 8.7. *Let $\mathcal{C}$ be the configuration of circles on S introduced in* 7.1*. Then there exists an embedding $H : S \to S'$ such that $H(a) = \rho(a)$ for every circle a of the configuration $\mathcal{C}$.*

Moreover, if S' is not a closed torus, then we can find an embedding H such that for every component c of ∂S its image $H(c)$ is either a component of $\partial S'$ or a nontrivial circle on S'.

Proof. Without any loss of generality we may restrict our attention to the case when ρ is defined on PMod_S.

We may assume that the configuration of circles $\rho(a)$, where a runs over the circles of $\mathcal{C}$, is in minimal position. We would like to apply Lemma 7.2 to the correspondence $a \mapsto \rho(a)$ in the role of $x \mapsto x'$. Note that this correspondence is injective by Lemma 8.1, satisfies the condition (i) from Lemma 7.2 by Lemma 8.2 and satisfies the condition (ii) from Lemma 7.2 by Lemma 8.5. Lemma 8.5 also ensures that S' is a closed surface of genus 2 if S is. Hence, the first part of Lemma 7.2 applies and there exists an embedding $H : S \to S'$ such that the image $H(S)$ contains all circles $\rho(a)$ for a in $\mathcal{C}$.

If S' is a closed torus, then S is a torus with one hole, because otherwise C contains at least two circles, contradicting to Lemmas 8.1 and 8.2. (Recall that S is not a closed torus.) In this case there are only two circles, a_1, a_2, in $\mathcal{C}$. The corresponding circles $\rho(a_1), \rho(a_2)$ intersect transversely at one point. This, clearly, implies the first assertion of the Lemma in this case. The second one is vacuos in this case.

Suppose now that S' is not a closed torus. Let us prove that for every component c of ∂S its image $H(c)$ does not bound a disc in S'. Suppose that to the contrary some of these images $H(c)$ bound discs in S'. Let R' be the result of adding these discs to $H(S)$. Note that because S' is not a closed torus, R' is not a closed torus also. Clearly, R' has the same genus as S and (strictly) less boundary components than S. Hence, the maximal number of circles in a system of circles on R' is less than the corresponding number for S (cf. 2.1; the fact that R' is not a closed torus is important here). It follows that some of the circles $\rho(a)$, where a runs over the components of C, are isotopic in R' and, hence, in S'. But this contradicts to Lemma 8.1. The contradiction proves our assertion.

It may happen that images $H(a)$ and $H(b)$ of two boundary components are isotopic. Then they are both nontrivial in S' and bound an annulus in S'. Let us choose one circle from each such pair and then add to them all nontrivial circles of the form $H(c)$, c is a boundary component, which are not isotopic to other circles of this form. The result is a system of circles on S', which we denote by D. Let δ be the corresponding simplex of $C(S')$. Let S'' be the component of S'_D containing $H(S)$. Obviously, S'' is diffeomorphic to S (because S'' can be obtained from $H(S)$ by glueing several annuli along some boundary components).

By Theorem 7.3, PMod_S is generated by Dehn twists t_a about circles a in $\mathcal{C}$. Hence, $\mathrm{Im}\,\rho$ is generated by Dehn twists $t_{\rho(a)} = \rho(t_a)$. All these Dehn twists and, hence, all elements of the image $\mathrm{Im}\,\rho$ can be represented by diffeomorphisms supported in $H(S)$ and, in particular, fixing D. It follows that $\mathrm{Im}\,\rho \subset M(\delta)$ and the composition $r_D \circ \rho : \mathrm{PMod}_S \to \mathrm{Mod}_{S'_D}$ is defined. Moreover, $\mathrm{Im}\, r_D \circ \rho \subset \mathrm{Mod}_{S'_D}(S'')$ and the composition $\pi_{S''} \circ r_D \circ \rho$ is defined (cf. 2.3 for notations). In addition, any element of $\mathrm{Im}\, r_D \circ \rho$ can be represented by a diffeomorphism equal to the identity on $S'_D \setminus S''$. It follows that $\mathrm{Ker}\,\pi_{S''} \circ r_D \circ \rho = \mathrm{Ker}\, r_D \circ \rho$.

We claim that this kernel is, in fact, trivial. Recall (cf. 2.3) that the kernel of r_D is the free abelian group generated by Dehn twists about components of D. It follows that $\mathrm{Ker}\, r_D \cap \mathrm{Im}\,\rho$ is a free abelian group contained in the center of $\mathrm{Im}\,\rho$. But, this center is finite in view of Lemma 6.2. Hence, the intersection $\mathrm{Ker}\, r_D \cap \mathrm{Im}\,\rho$ is free abelian and finite at the same time, so it has to be trivial. Hence, $\mathrm{Ker}\, r_D \cap \mathrm{Im}\,\rho = \{1\}$ and $\mathrm{Ker}\, r_D \circ \rho = \{1\}$. It follows that $\mathrm{Ker}\,\pi_{S''} \circ r_D \circ \rho = \{1\}$ and the homomorphism $\rho'' = \pi_{S''} \circ r_D \circ \rho$ is injective.

Now, we can apply previous results to $\rho'' : \mathrm{PMod}_S \to \mathrm{Mod}_{S''}$ in the role of ρ (clearly, ρ'' is twist-preserving). Since S'' is diffeomorphic to S and the condition (iii) of Lemma 7.2 is fulfilled by Lemma 8.6, now we are in the position to apply the second part of the Lemma 7.2. It gives us an emebdding $H_0 : S \to S''$ such that $H_0(a) = \rho(a)$ for all a in $\mathcal{C}$. After deforming, if necessary, H_0 a little we may assume that $H_0(S) \subset \mathrm{int}\, S''$. Then the composition H of H_0 with the canonical map $S'' \to S'$ is also an embedding and if the deformation is small enough, then $H(a) = \rho(a)$ for all a in $\mathcal{C}$. Applying to H the already proved results, we see that the image $H(c)$ of any boundary component c does not bound a disc in S'. This means that if $H(c)$ is a trivial circle for a boundary component c, than $H(c)$ is parallel to a component of $\partial S'$. By deforming H in a neighborhood of such boundary components c we now can fullfil the second condition of the Lemma. This completes the proof. □

Lemma 8.8. *If $H : S \to S'$ is a diffeomorphism such that $\rho(a) = H(a)$ for all circles a in $\mathcal{C}$, then ρ is induced by H.*

Proof. Recall that the isomorphism $H_* : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ induced by H is defined by the formula $H_*[G] = [HGH^{-1}]$, where G is an orientation preserving diffeomorphism $S \to S$ and, as before, $[F]$ denotes the isotopy class of a diffeomorphism F. Obviously, $H_*(\mathrm{PMod}_S) = \mathrm{PMod}_{S'}$.

By Theorem 7.3, the Dehn twists t_a over the circles a of $\mathcal{C}$ form a set of generators of PMod_S. Since $\rho(t_a) = t_{\rho(a)} = t_{H(a)} = H_*(t_a)$ for all such a, this implies that ρ agrees with H_* on PMod_S. In particular, this proves the Lemma if ρ is defined on PMod_S.

It remains to consider the case where ρ is defined on Mod_S. Let $\sigma = H_*^{-1} \circ \rho : \mathrm{Mod}_S \to \mathrm{Mod}_S$. Then σ is equal to the identity on PMod_S. Recall that if $f \in \mathrm{Mod}_S$ and $\alpha \in V(S)$, then $f t_\alpha f^{-1} = t_{f(\alpha)}$. By applying σ to this equality, we get $\sigma(f)\sigma(t_\alpha)\sigma(f)^{-1} = \sigma(t_{f(\alpha)})$ and, since, $t_\alpha, t_{f(\alpha)} \in \mathrm{PMod}_S$, this implies that $\sigma(f) t_\alpha \sigma(f)^{-1} = t_{f(\alpha)}$ and, consequently, $t_{\sigma(f)(\alpha)} = t_{f(\alpha)}$. In view of Theorem 4.1, this in turn implies that $\sigma(f)(\alpha) = f(\alpha)$ or $\sigma(f)^{-1} f(\alpha) = \alpha$ for all $\alpha \in V(S)$. Now it follows from Lemma 6.1 that $\sigma(f)^{-1} f \in C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$. Recall that S is assumed to be of positive genus and not a closed torus. If S is, in addition, not a torus with one or two holes, then the centralizer $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ is trivial by Theorem 6.3. Hence, $\sigma(f)^{-1} f = 1$ for all f and $\sigma = \mathrm{id}$ in this case. If S is a torus with one or two holes, then $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) \cong \mathbb{Z}/2\mathbb{Z}$, by the same Theorem 6.3. If S is a torus with one hole, then $\mathrm{Mod}_S = \mathrm{PMod}_S$ and, hence, $\sigma = \mathrm{id}$. Finally, if S is torus with two holes, then PMod_S is a subgroup of index 2 in Mod_S. In this case, if $f \notin \mathrm{PMod}_S$, then $\sigma(f) \notin \mathrm{PMod}_S$, and, since PMod_S is of index 2 in Mod_S, it follows that $\sigma(f)^{-1} f \in \mathrm{PMod}_S$. Combining this with the fact that $\sigma(f)^{-1} f \in C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$, proved above, we conclude that $\sigma(f)^{-1} f \in C(\mathrm{PMod}_S)$. But $C(\mathrm{PMod}_S)$ is trivial in this case by Theorem 6.3. It follows that $\sigma(f)^{-1} f = 1$ for all f and $\sigma = \mathrm{id}$ in this case also. So, $\sigma = \mathrm{id}$ in all cases. It follows that $\rho = H_*$. This completes the proof. □

Theorem 8.9. *Suppose that S is a surface of genus $g \geq 2$. If $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective twist-preserving homomorphism, then ρ is induced by a diffeomorphism $S \to S'$.*

Proof. It is especially simple (after all previous work) for closed surfaces. If S is closed, then the embedding H provided by Lemma 8.7 is a diffeomorphism. Hence, Lemma 8.8 implies that ρ is induced by H. Note that we need only the trivial part of (the proof of) Lemma 8.8 here, because $\mathrm{PMod}_S = \mathrm{Mod}_S$ in this case.

Let us consider now the general case. We will use the notations of Section 5.2. Let $H : S \to S'$ be the embedding provided by Lemma 8.7. If $F : S \to S$ is a diffeomorphism fixed on ∂S, then we can define a diffeomorphism $S' \to S'$ by extending the diffeomorphism $H \circ F \circ H^{-1} : H(S) \to H(S)$ by the identity to the whole surface S'. By passing to the isotopy classes, we get the (well known) homomorphism $\mathcal{M}_S \to \mathcal{M}_{S'}$ induced by H. We will denote it by H_*. Let us consider the followng diagram.

$$\begin{array}{ccc} \mathcal{M}_S & \xrightarrow{H_*} & \mathcal{M}_{S'} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ \mathrm{PMod}_S & \xrightarrow{\rho} & \mathrm{Mod}_{S'} \end{array}$$

The vertical maps are the canonical homomorphisms $p : \mathcal{M}_S \to \mathrm{PMod}_S$, $p' : \mathcal{M}_{S'} \to \mathrm{Mod}_{S'}$. According to Lemma 8.7, $\rho \circ p(\tilde{t}_a) = \rho(t_a) = t_{H(a)}$ for all a in $\mathcal{C}$. Also, clearly, $p' \circ H_*(\tilde{t}_a) = p'(\tilde{t}_{H(a)}) = t_{H(a)}$ for a in $\mathcal{C}$. It follows that $\rho \circ p$ and $p' \circ H_*$ agree on the set $\{\tilde{t}_a : a \in \mathcal{C}\}$. But, in view of Theorems 7.3 and 5.3, this set generates $\mathcal{M}_S$. (This reference to Theorem 5.3 is the only place in the proof where the assumption $g \geq 2$ is used.) Hence, our diagram is commutative.

Now, if c is a boundary component of S such that $H(c)$ is a nontrivial circle on S', then $p' \circ H_*(\tilde{t}_c) = p'(\tilde{t}_{H(c)}) = t_{H(c)} \neq 1$. On the other hand, $\rho \circ p(\tilde{t}_c) = \rho(1) = 1$. The contradiction shows that $H(c)$ cannot be a nontrivial circle for a boundary component c. In view of Lemma 8.7, this means that $H(\partial S) \subset \partial S'$ (note that since S is of genus at least 2 and, hence, PMod_S contains a free abelian subgroup of rank 3, S' cannot be a closed torus). It follows that H is diffeomorphism. An application of Lemma 8.8 completes the proof. □

Lemma 8.10. *Let g, b (respectively, g', b') be the genus and the number of boundary components of S (respectively, S'). Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. Then either S is a torus with one hole and S' is a closed torus, or $3g + b \leq 3g' + b' \leq 3g + b + 1$.*

Proof. Since, by our assumptions, S is of positive genus and not a closed torus, the maxima of ranks of abelian subgroups of Mod_S is equal to $3g - 3 + b$ (cf. 2.1). Note that $i(a, b) = 1$ for some circles a, b on S. By Lemma 8.2 $i(\rho(a), \rho(b)) = 1$ and, hence, S' is of positive genus. It follows that the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is equal to $3g' - 3 + b'$ unless S' is a closed torus. Also, $3g - 3 + b \geq 2$, unless S is a torus with one hole (when it is equal to 1). Hence, if S is not a torus with one hole, the injectivity $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ implies that the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is also at least 2 and S' cannot be a closed torus. Now, it is clear that if S is not a torus with one hole and S' is not a closed torus, then the first inequality follows from the injectivity of ρ again and the second one from the assumption that the maxima of ranks differ by at most 1. □

Lemma 8.11. *Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. If S is a torus with one hole, suppose that, in addition, S' is not a closed torus. Let $H : S \to S'$ be an embedding as in Lemma 8.7. If*

$H(S) \neq S'$, then there exists a unique isotopy class $\gamma \in V(S')$ such that for each component c of ∂S, either $H(c)$ is a component of $\partial S'$ or $H(c) \in \gamma$.

Proof. We use the notations g, b, g', b' from Lemma 8.10. Since $H(S) \neq S'$, there exists a component c of ∂S such that $H(c)$ is a nontrivial circle on S'. If a_1, a_2 are two components of ∂S such that $H(a_1), H(a_2)$ are nonisotopic nontrivial circles on S', then $H(C) \cup H(a_1) \cup H(a_2)$ is a system of circles on S' consisting of $3g - 1 + b$ components. In view of 2.1, $3g - 1 + b \leq 3g' - 3 + b'$. Since $3g' + b' \leq 3g + b + 1$ by Lemma 8.10, this is impossible. The Lemma follows. □

Lemma 8.12. *In the situation of Lemma 8.11, if $H(S) \neq S'$ and γ is the isotopy class provided by Lemma 8.11, then the image of ρ is contained in the stabilizer of γ in $\mathrm{Mod}_{S'}$.*

Proof. We use the notations g, b, g', b' from Lemma 8.10. Since $H(S) \neq S'$, there exists a component c of ∂S such that $H(c)$ is a nontrivial circle on S', and γ is the isotopy class of $H(c)$. Since $H : S \to S'$ is an embedding, $H(c)$ is disjoint from $H(a)$ for every nontrivial circle a on S. If a is a circle of the configuration $\mathcal{C}$, then, in view of Lemma 8.7, $\rho(a) = H(a)$ and, hence, $\rho(t_a) = t_{\rho(a)} = t_{H(a)}$ commutes with $t_{H(c)} = t_\gamma$. Now, Theorem 7.3 implies that the group $\rho(\mathrm{PMod}_S)$ commutes with t_γ. If ρ is defined on PMod_S, it remains only to apply Theorem 4.1 and the standard relation $f t_\gamma f^{-1} = t_{f(\gamma)}$.

If ρ is defined on Mod_S, an additional argument is needed (compare the proof of Lemma 8.8). Let $f \in \mathrm{Mod}_S$, $f' = \rho(f)$ and $\beta = f'(\gamma)$. If a is a nonseparating circle on S and α is its isotopy class, then $f^{-1} t_\alpha f \in \mathrm{PMod}_S$ and, hence, t_γ commutes with $\rho(f^{-1} t_\alpha f) = (f')^{-1} \rho(t_\alpha) f' = (f')^{-1} t_{\rho(\alpha)} f' = t_{(f')^{-1}(\rho(\alpha))}$. By Theorem 4.2, this implies that $i(\gamma, (f')^{-1}(\rho(\alpha))) = 0$. Let b be some circle in the isotopy class $\beta = f'(\gamma)$. Then $i(b, H(a)) = i(b, \rho(a)) = i(\beta, \rho(\alpha)) = i(f'(\gamma), \rho(\alpha)) = i(\gamma, (f')^{-1}(\rho(\alpha))) = 0$. Since $i(b, H(a)) = 0$ for all nonseparating circles a on S (in particular, for all circles in $\mathcal{C}$), we can replace b by an isotopic circle disjoint from $H(S)$.

If b is not isotopic to $H(c)$, then $H(C) \cup H(c) \cup b$ is system of circles on S' with $3g - 1 + b$ components. Hence, $3g - 1 + b \leq 3g' - 3 + b'$. Since $3g' + b' \leq 3g + b + 1$ by Lemma 8.10, this is impossible. Hence, b is isotopic to $H(c)$ and $\rho(f)(\gamma) = f'(\gamma) = \beta = \gamma$. Since $f \in \mathrm{Mod}_S$ is arbitrary, this completes the proof. □

Lemma 8.13. *In the situation of Lemma 8.11, if $H(S) \neq S'$ and γ is the isotopy class provided by Lemma 8.11, then for each nontrivial circle a on S there exists an integer N_a such that $\rho(t_a) = t_{H(a)} t_\alpha^{N_a}$. If $b = F(a)$ for some diffeomorphism $F : S \to S$, then $N_b = N_a$.*

Proof. As in the proof of Theorem 8.9, let us consider the following diagram.

$$\begin{array}{ccc} \mathcal{M}_S & \xrightarrow{H_*} & \mathcal{M}_{S'} \\ \Big\downarrow p & & \Big\downarrow p' \\ \mathrm{PMod}_S & \xrightarrow{\rho} & \mathrm{Mod}_{S'} \end{array}$$

Recall that p, p' are the canonical homomorphisms and H_* maps the isotopy class of a diffeomorphism $F : S \to S$ into the isotopy class of the extension of $H \circ F \circ H^{-1}$ by the identity to the whole surface S'. Since now the genus of S may be equal to 1, we cannot invoke the Theorem 5.3 in order to conclude that this diagram is commutative. But we still know that $\rho \circ p$ is equal to $p' \circ H_*$ on a Dehn twist t_a if a is a circle of the configuration $\mathcal{C}$.

Let a be a nontrivial circle on S. In view of Theorems 5.1 and 7.3, we have

$$\tilde{t}_a = \tilde{t}_{u_1}\tilde{t}_{u_2}\ldots\tilde{t}_{u_m}\tilde{t}_{v_1}\tilde{t}_{v_2}\ldots\tilde{t}_{v_n}$$

for some circles $u_1, u_2, \ldots, u_m \in \mathcal{C}$ and some boundary components $v_1, v_2, \ldots, v_n$ of S. Since $p(\tilde{t}_{u_i}) = t_{u_i}$ and $p(\tilde{t}_{v_j}) = 1$ for all i, j, we have

$$\begin{aligned} \rho(t_a) &= \rho \circ p(\tilde{t}_a) \\ &= \rho(t_{u_1})\rho(t_{u_2})\ldots\rho(t_{u_m}) \\ &= t_{\rho(u_1)}t_{\rho(u_2)}\ldots t_{\rho(u_m)} \\ &= t_{H(u_1)}t_{H(u_2)}\ldots t_{H(u_m)} \end{aligned} \tag{8.1}$$

(the last equality follows from Lemma 8.7). On the other hand,

$$\tilde{t}_{H(a)} = H_*(\tilde{t}_a) = \tilde{t}_{H(u_1)}\tilde{t}_{H(u_2)}\ldots\tilde{t}_{H(u_m)}\tilde{t}_{H(v_1)}\tilde{t}_{H(v_2)}\ldots\tilde{t}_{H(v_n)}. \tag{8.2}$$

By the definition of γ, each of the circles $H(v_i)$ is either a component of $\partial S'$ (and then $p'(\tilde{t}_{H(v_i)}) = 1$), or in the isotopy class γ (and then $p'(\tilde{t}_{H(v_i)}) = t_\gamma$). It follows that

$$t_{H(a)} = p'(\tilde{t}_{H(a)}) = t_{H(u_1)}t_{H(u_2)}\ldots t_{H(u_m)}t_\gamma^{-N_a} \tag{8.3}$$

for some integer number N_a. By combyning (8.1) and (8.3), we get

$$\rho(t_a)t_\gamma^{-N_a} = t_{H(a)}$$

and

$$\rho(t_a) = t_{H(a)}t_\gamma^{N_a}.$$

This proves the first assertion of the Lemma.

Suppose that for two nontrivial circles a and b on S there exists a diffeomorphism $F : S \to S$ such that $F(a) = b$. We may assume that F is orientation-preserving. Let $f \in \mathrm{Mod}_S$ be the isotopy class of F and $f' = \rho(f)$. Then $t_b = f t_a f^{-1}$ and, hence, $\rho(t_b) = f'\rho(t_a)(f')^{-1}$. Since $\rho(\mathrm{Mod}_S)$ is contained in the stabilizer of γ in

$\mathrm{Mod}_{S'}$, $f'(\gamma) = \gamma$. Since $H(c) \in \gamma$, we may choose a diffeomorphism $F' \in f'$ such that $F'(H(c)) = H(c)$. By the already proved first assertion of the Lemma, $\rho(t_b) = f'\rho(t_a)(f')^{-1}$ implies that

$$t_{H(b)}t_\gamma^{N_b} = f' t_{H(a)}t_\gamma^{N_a}(f')^{-1} = t_{F'(H(a))}t_{f'(\gamma)}^{N_a} = t_{F'(H(a))}t_\gamma^{N_a}.$$

Note that $H(a)$ and $H(b)$ are not isotopic on S' to $H(c)$. Thus $F'(H(a))$ and $F'(H(b))$ are not isotopic to $H(c)$.

Clearly, if $N_b = 0$, then $H(b)$ is a realization of the canonical reduction system for $t_{H(b)}t_\gamma^{N_b}$. Otherwise, $H(b) \cup H(c)$ is a realization. Likewise, if $N_a = 0$, then $F'(H(a))$ is a realization of the canonical reduction system for $t_{F'(H(a))}t_\gamma^{N_a}$. Otherwise, $F'(H(a)) \cup H(c)$ is a realization. Since $t_{H(b)}t_\gamma^{N_b} = t_{F'(H(a))}t_\gamma^{N_a}$, we conclude that:

(i) $F'(H(a))$ is isotopic to $H(b)$;

(ii) $N_a = 0$ if and only if $N_b = 0$.

Since $t_{H(b)}t_\gamma^{N_b} = t_{F'(H(a))}t_\gamma^{N_a}$, (i) implies that $t_\alpha^{N_b} = t_\alpha^{N_a}$. Then, (ii) and Theorem 4.1 imply that $N_a = N_b$. This completes the proof of the second assertion of the Lemma. □

Lemma 8.14. *If a is a circle on S bounding a torus with one hole in S, then $\rho(t_a)$ is a Dehn twist. In the situation of Lemma* 8.11, $\rho(t_a) = t_{H(a)}$.

Proof. Let P be the torus with one hole bounded in S by a. Clearly, there are two circles u, v in P intersecting transversely in one point. By Theorem 4.3 $t_a = (t_u t_v)^6$. Hence, $\rho(t_a) = (\rho(t_u)\rho(t_v))^6 = (t_{\rho(u)}t_{\rho(v)})^6$ (note that both u and v are nonseparating). By Lemma 8.2 $i(\rho(u), \rho(v)) = 1$ and, hence, we may assume that $\rho(u)$ and $\rho(v)$ intersect transversely at one point. Now, Theorem 4.3 implies that $(t_{\rho(u)}t_{\rho(v)})^6 = t_{a'}$ for some circle a' on S'. It follows that $\rho(t_a) = t_{a'}$ is a Dehn twist. This proves the first assertion of the Lemma.

Clearly, the first assertion toghether with Lemma 8.13 imply the second assertion. This completes the proof. □

Theorem 8.15. *Suppose that S has positive genus and is not a closed torus. If S is a torus with one hole suppose, in addition, that S' is not a closed torus. Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. If $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective twist-preserving homomorphism, then ρ is induced by a diffeomorphism $S \to S'$.*

Proof. If the genus of S is at least 2, Theorem 8.9 applies. Hence, we need to consider only the case when the genus of the surface S is 1.

Let $H : S \to S'$ be an embedding as in Lemma 8.7. In view of Lemma 8.8, we may restrict our attention to the case when $H(S) \neq S'$. Let $\gamma \in V(S')$ the isotopy class provided by Lemma 8.11. In view of Lemmas 8.7 and 8.11, γ is the isotopy class of the image $H(c)$ of some boundary component c of S. By Lemma 8.13, for

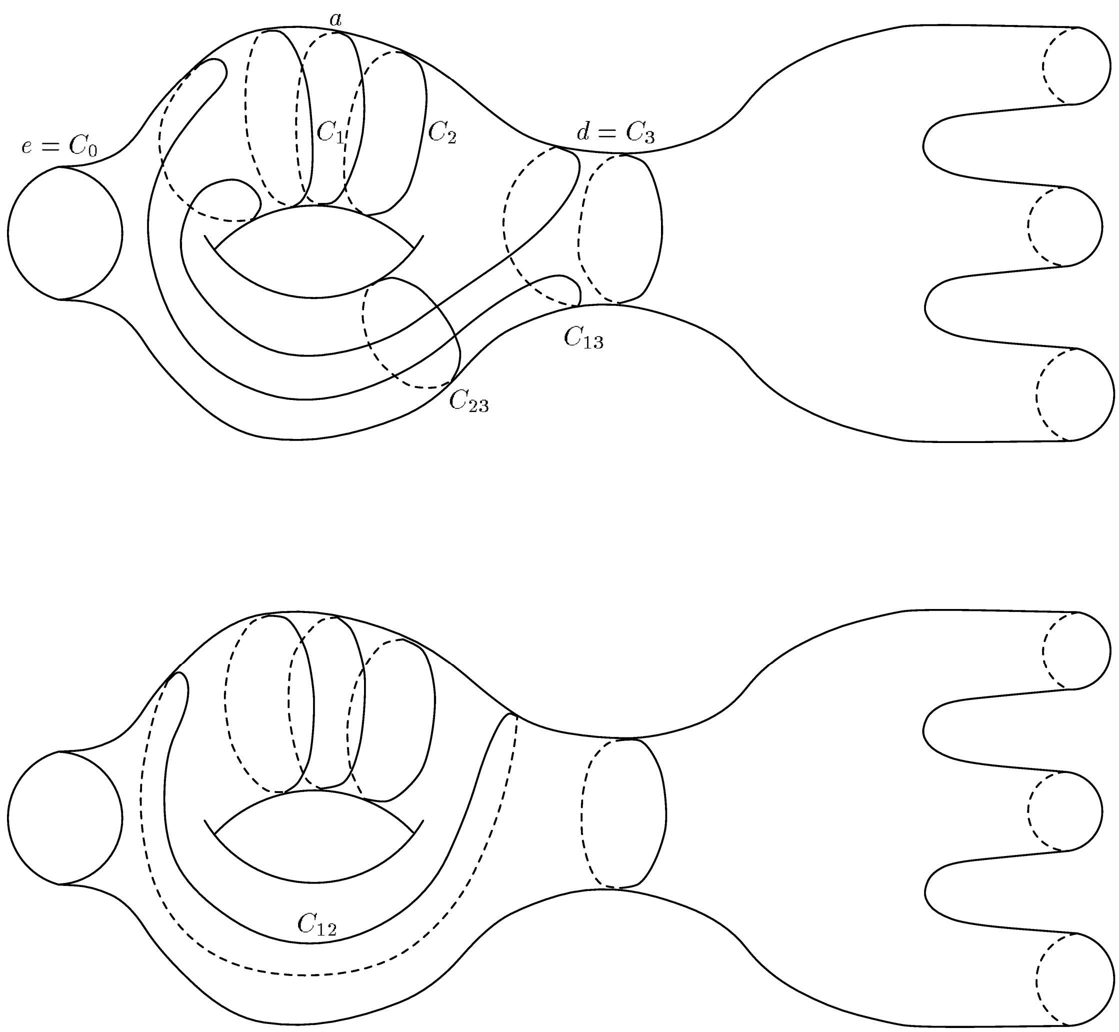


Figure 8.8

each nontrivial circle a on S, there exists an integer N_a such that $\rho(t_a) = t_{H(a)} t_\alpha^{N_a}$. Of course, since ρ is twist-preserving, $N_a = 0$ for each nonseparating circle a on S.

We will divide the following arguments into three cases, according to the number of boundary components of S (remember that S is assumed to be of genus 1 now).

The first case is when there are at least 3 boundary components. Let d be some circle separating S into a torus two holes P and a sphere Q with the same number of holes as S. The boundary ∂P consists of d and a component e of ∂S. The boundary ∂Q consists of d and the components of $\partial S \setminus e$. Let a be a nonseparating circle on P and let S_1 be the complement in P of an (open) annulus having a as its axis. Then S_1 is a sphere with four holes. We can identify S_1 with the sphere with four holes S_0 from 5.2 in such a way that: (i) $C_0 = e$; (ii) C_1 and C_2 are isotopic to a on P; (iii) $C_3 = d$. Then C_1, C_2, C_{13} and C_{23} are nonseparating circles on S and C_{12} divides S into a torus with one hole and a sphere with holes. See Figure 8.8.

Let t_i, t_{ij} be the images in Mod_S of the elements $\tilde{t}_i, \tilde{t}_{ij}$ of the group $\mathcal{M}_S$ introduced in 5.2 respectively. Since $C_0 = e$ is a boundary component, $t_0 = 1$. Hence, the lantern relation (5.1) implies that

$$t_1 t_2 t_3 = t_{12} t_{13} t_{23}. \tag{8.4}$$

Hence

$$\rho(t_1)\rho(t_2)\rho(t_3) = \rho(t_{12})\rho(t_{13})\rho(t_{23}). \tag{8.5}$$

Since the circles C_1, C_2, C_{13} and C_{23} are nonseparating, we have

$$\begin{aligned} &\rho(t_1) = t_{H(C_1)}; \qquad \rho(t_2) = t_{H(C_2)};\\ &\rho(t_{13}) = t_{H(C_{13})}; \qquad \rho(t_{23}) = t_{H(C_{23})}. \end{aligned} \tag{8.6}$$

Since C_{12} bounds a disc with one hole in S, Lemma 8.14 implies that also

$$\rho(t_{12}) = t_{H(C_{12})}. \tag{8.7}$$

Also, by Lemma 8.13,

$$\rho(t_3) = t_{H(C_3)} t_\gamma^N \tag{8.8}$$

for some integer N. It follows from (8.5) and (8.6)–(8.8) that

$$t_{H(C_1)} t_{H(C_2)} t_{H(C_3)} t_\gamma^N = t_{H(C_{12})} t_{H(C_{13})} t_{H(C_{23})}. \tag{8.9}$$

Now, it follows from Lemma 8.11 that at most two components of ∂S are mapped by H to nontrivial circles on S'. Since we assume now that S has at least 3 boundary components, we may choose the circle d in such a way that the circle e is mapped by H to a boundary component of S'. By applying the lantern relation (5.1) to the sphere with four holes $H(S_1)$ and the circles $H(C_i)$, $H(C_{ij})$, we get, in the same way as (8.4), the following relation

$$t_{H(C_1)} t_{H(C_2)} t_{H(C_3)} = t_{H(C_{12})} t_{H(C_{13})} t_{H(C_{23})}. \tag{8.10}$$

Comparing (8.9) and (8.10), we conclude that $t_\gamma^N = 1$ and, hence, $N = 0$. In other words, $\rho(t_d) = t_{H(d)}$ or, what is the same, $\rho(t_3) = t_{H(C_3)}$.

On the other hand, we may choose the circle d in such a way that e will be equal to c. Then all the circles C_i, C_{ij} will be mapped by H to nontrivial circles on S'. By applying the lantern relation (5.1) to the sphere with four holes $H(S_1)$ and the circles $H(C_i), H(C_{ij})$, we get now

$$t_{H(C_0)}t_{H(C_1)}t_{H(C_2)}t_{H(C_3)} = t_{H(C_{12})}t_{H(C_{13})}t_{H(C_{23})}. \tag{8.11}$$

But, the second assertion of Lemma 8.13 implies that the number N from (8.8) is the same for all choices of d and, hence, is equal to 0 in this case also. This means that

$$\rho(t_3) = t_{H(C_3)} \tag{8.12}$$

and (8.9) holds with $N = 0$, i.e. (8.10) holds. Note that this time we deduced (8.10) not from the lantern relation on $H(S_1)$, but from (8.5), and the result contradicts to the lantern relation (8.11) on $H(S_1)$ because $H(C_0) = H(e) = H(c)$ is now a nontrivial circle on S' (its isotopy class is γ). The contradiction shows that if $H(S) \neq S'$, then S cannot be a torus with at least three holes.

The second case is that of a torus with two holes. In this case we are going to use the surface S itself in the same way as we used the surface P in the first case. Now the boundary ∂S consists of c and another component e. Let a be a nonseparating circle on S and let S_1 be the complement in S of an (open) annulus having a as its axis. Again, S_1 is a sphere with four holes. We can identify S_1 with the sphere with four holes S_0 from 5.2 in such a way that: (i) $C_0 = c$; (ii) C_1 and C_2 are isotopic to a on S; (iii) $C_3 = e$. As in the first case, C_1, C_2, C_{13} and C_{23} are nonseparating circles on S and C_{12} divides S into a torus with one hole and a sphere with, this time three, holes.

Again, let t_i, t_{ij} be the images in Mod_S of the elements $\tilde{t}_i, \tilde{t}_{ij}$ of the group $\mathcal{M}_S$ introduced in 5.2. This time both C_0 and C_3 are boundary components, and, hence, $t_0 = t_3 = 1$. The lantern relation (5.1) now implies that

$$t_1 t_2 = t_{12}t_{13}t_{23}. \tag{8.13}$$

and

$$\rho(t_1)\rho(t_2) = \rho(t_{12})\rho(t_{13})\rho(t_{23}). \tag{8.14}$$

By the same reasons as before, the relations (8.6) and (8.7) hold. By substituting (8.6) and (8.7) into (8.14), we get

$$t_{H(C_1)}t_{H(C_2)} = t_{H(C_{12})}t_{H(C_{13})}t_{H(C_{23})}. \tag{8.15}$$

Now, we apply the lantern realation (5.1) to the sphere with four holes $H(S_1)$ and the circles $H(C_i), H(C_{ij})$. Note that the circle $H(C_0)$ is in the isotopy class γ and the circles $H(C_1), H(C_2), H(C_{12}), H(C_{13}), H(C_{23})$ are all nontrivial on S'. The circle

$H(C_3)$ is either a boundary component or in the isotopy class γ in view of Lemma 8.11. We get

$$t_\gamma t_{H(C_1)} t_{H(C_2)} = t_{H(C_{12})} t_{H(C_{13})} t_{H(C_{23})} \tag{8.16}$$

if $H(C_3)$ is a boundary component and

$$t_\gamma t_{H(C_1)} t_{H(C_2)} t_\gamma = t_{H(C_{12})} t_{H(C_{13})} t_{H(C_{23})} \tag{8.17}$$

if $H(C_3)$ is in the isotopy class γ. Comparing (8.15) with (8.16) or (8.17) we conclude that either t_γ or t_γ^2 is equal to 1. This contradicts to the fact that γ is the isotopy class of the nontrivial circle $H(c)$. The contradiction shows that if $H(S) \neq S'$, then S cannot be a torus with two holes.

The third and the final case is that of a torus with one hole. In this case the only boundary component c of S is mapped by H to a nontrivial circle on S' (in the isotopy class γ). By applying Lemma 8.14 to c in the role of a, we conclude that $\rho(t_c) = t_{H(c)} = t_\gamma$. But $t_c = 1$, because c is a boundary component, and $t_\gamma \neq 1$, because $H(c)$ is nontrivial. The contradiction shows that it $H(S) \neq S'$, then S cannot be a torus with one hole either.

Hence, the assumption $H(S) \neq S'$ leads to a contradiction in all the cases and Theorem follows from Lemma 8.8. □

8.16. Remark. If S is a torus with one hole and S' is a closed torus, then any injective twist preserving homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an isomorphism induced, in a natural sense, by an embedding $H : S \to S'$. We leave the (easy) proof to the reader.

9. Systems of Separating Circles

In this section, S denotes a compact connected orientable surface of genus g with b boundary components. We call two circles a, b on S *topologically equivalent* if there is a diffeomorphism $F : S \to S$ such that $F(a) = b$. The goal of this section is to show that the maximal or "almost" maximal systems of *separating* circles on S with all components topologically equivalent are, in fact, very special. The main features of such systems of circles are described in Theorems 9.1 and 9.2.

Theorem 9.1. *Let S be a compact connected orientable surface of genus g with b boundary components. Let C be a system of topologically equivalent separating circles on S. Suppose that C has $3g - 4 + b$ components. Then for each component a of C there exists a disc with two holes P_a embedded in S such that ∂P_a consists of a and two components of ∂S. Moreover, S is either a sphere with five, six, seven or eight holes or a torus with two holes.*

Proof. Let $n = 3g - 4 + b$. Since $n \geq 1$, $3g + b \geq 5$. Thus S is not a a sphere with at most four holes or a torus with at most one hole.

Let R be the surface obtained by cutting S along C. Since each component of C is a separating circle on S, the surface R has exactly $n+1$ components. Moreover, the genus of S is the sum of the genera of the components of R. Hence, there must be a system D of g nonseparating circles on R. The union $C \cup D$ is a system of circles on S with $n+g$ components. Hence, $n+g \leq 3g-3+b$. Since $3g-4+b=n$, we conclude that one of the following conditions must hold:

(i) $g=1$ and $n=b-1$,

(ii) $g=0$ and $n=b-4$.

For each component Q of R, let C_Q be a maximal system of circles on Q and m_Q be the number of components of C_Q. Let m be the sum of m_Q over all components Q of R. The union of C and the systems of circles C_Q over the components Q of R is a maximal system of circles on S. Hence, $3g-3+b=n+m$. Since $n=3g-4+b$, we conclude that $m \leq 1$. Thus, every component of R is either a disc with two holes, a sphere with four holes or a torus with one hole. Moreover, there is at most one component of R which is not a disc with two holes. Since R has $n+1$ components, at least one component of R is a disc with two holes.

Consider the case $g=1$ and $n=b-1$ first. Since the genus of S is equal to the sum of the genera of the components of R, exactly one component Q of R has the genus one. By the preceding considerations, Q is a torus with one hole. Since S is not a torus with one hole, ∂Q must correspond to a component a of C. Thus a is a circle on S bounding a torus with one hole embedded in S. Note that every nontrivial circle on Q is nonseparating. It follows that $\operatorname{int} Q$ does not contain any components of C and, hence, Q is a component of R. Since all components of C are topologically equivalent, it follows that R has at least n components which are tori with one hole. Since R has at most one component which is not a disc with two holes, $n=1$ and C consists of a single separating circle a on S. Thus, R has two components, P and Q, which meet along a. Since Q is a torus with one hole and R has at most one component which is not a disc with two holes, P is a disc with two holes. Hence, S is a torus with two holes. Moreover, P is embedded in S in such a way that ∂P consists of a and the two components of ∂S.

Let us consider now the case $g=0$ and $n=b-4$. Let a be a component of C. The circle a separates S into two spheres with holes P_a and Q_a. Let $p+1$ and $q+1$ be the numbers of boundary components of P_a and Q_a respectively. Since a is nontrivial, $p, q \geq 2$. We may assume that $p \leq q$. Since the components of C are topologically equivalent circles on S, the pair of integers p, q does not depend upon the component a of C. Moreover, $p+q=b$.

Suppose that $n=1$. Since $n=b-4$, b is equal to 5. Hence, S is a sphere with five holes. The system C consists of a single separating circle a. Since $p, q \geq 2$ and and $p \leq q$, we have $p=2$ and $q=3$. Hence, P_a is a discs with two holes embedded in S such that ∂P_a consists of a and two components of ∂S.

Suppose now that $n \geq 2$. Let a and b be distinct components of C. If b is contained in P_a, then either P_b or Q_b is contained in P_a. Since both P_b and Q_b contain at least as many boundary components of S as P_a (namely, p or $q \geq p$), this implies that b is isotopic to a. Contradiction with the fact that C is a system of circles implies that b is contained in Q_a. A similar argument implies that P_b is contained in Q_a and $q > p$ (if $q = p$, then b is isotopic to a again). It follows that P_a and P_b are disjoint for every pair of distinct components a, b of C. Hence R is the union of a component Q_0 and the components P_a over the components a of C. Each component of ∂Q_0 is either a component of C or a component of ∂S. Moreover, Q_0 and P_a meet along a for every component a of C. Thus, no two components of ∂Q_0 correspond to the same component a of C. Hence, Q_0 is embedded in S. Since at most one component of R is not a disc with two holes and $n \geq 2$, P_a is a disc with two holes for each component a of C. Since the genus of S is zero, Q_0 is not a torus with one hole. Hence, Q_0 is either a disc with two holes or a sphere with four holes.

Thus $2 \leq n \leq 4$. Since $n = b - 4$, S is a sphere with six, seven or eight holes. This completes the proof. □

Theorem 9.2. *Let S be a compact connected orientable surface of genus g with b boundary components. Let C be a system of topologically equivalent separating circles on S. Suppose that C has $3g - 3 + b$ components. Then for each component a of C there exists a disc with two holes P_a embedded in S such that ∂P_a consists of a and two components of ∂S. Moreover, S is a sphere with four, five or six holes.*

Proof. Let $n = 3g - 3 + b$. Since $n \geq 1$, S is not a disc with at most two holes or a torus with at most one hole.

Let R be the surface obtained by cutting S along C. Since each component of C is a separating circle on S, R has exactly $n + 1$ components. Moreover, the genus of S is the sum of the genera of the components of R. Hence, there must be a system D of g nonseparating circles on R. The union $C \cup D$ is a system of circles on S with $n + g$ components. Hence, $n + g \leq 3g - 3 + b$. Since $n = 3g - 3 + b$, we conclude that $g = 0$ and $n = b - 3$.

Since $n = 3g - 3 + b$, C is a maximal system of circles on S. Hence, each component of R is a disc with two holes.

Let a be a component of C. The circle a separates S into two spheres with holes P_a and Q_a. Let $p + 1$ and $q + 1$ be the numbers of boundary components of P_a and Q_a respectively. Since a is nontrivial, $p, q \geq 2$. We may assume that $p \leq q$. Since the components of C are topologically equivalent circles on S, the pair of integers p, q does not depend upon the component a of C. Moreover, $p + q = b$.

Suppose that $n = 1$. Since $n = b - 3$, b is equal to 4. In this case, S is a sphere with four holes and C consists of a single separating circle a. Both P_a and Q_a are discs with two holes embedded in S such that their boundaries consist of a and two components of ∂S.

Suppose that $n \geq 2$. Exactly the same argument as in the proof of Theorem 9.1 now implies that P_a and P_b are disjoint for every pair of distinct components a, b of C. Hence, R is the union of a component Q_0 and the components P_a over the components a of C. Each component of ∂Q_0 is either a component of C or a component of ∂S. Moreover, Q_0 and P_a meet along a for every component a of C. Thus, no two components of ∂Q_0 correspond to the same component a of C. Hence, Q_0 is embedded in S. Since each component of R is a disc with two holes, P_a is a disc with two holes for each component a of C and Q_0 is a disc with two holes.

Hence, $2 \leq n \leq 3$. Since $n = b - 3$, S is a sphere with five or six holes. This completes the proof. □

10. Almost twist-preserving homomorphisms

As in Section 8, S and S' denote compact connected *oriented* surfaces. We assume that S has positive genus and is not a closed torus. Let $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ be an injective homomorphism. We say that ρ is *almost twist-preserving* if, for each isotopy class $\alpha \in V_0(S)$ there exists an isotopy class $\rho(\alpha) \in V(S')$ and nonzero integers M and N such that $\rho(t_\alpha^M) = t_{\rho(\alpha)}^N$. The goal of this section is to prove that, with few exceptions, injective almost twist-preserving homomorphisms are actually induced by diffeomorphisms $S \to S'$ if the maxima of ranks of Abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by at most one. See Theorems 10.8, 10.9 and 10.10 for the exact statements.

For the remainder of this section, we assume that $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective almost twist-preserving homomorphism. By Theorem 4.1, the isotopy class $\rho(\alpha) \in V(S')$ is uniquely determined by the equality $\rho(t_\alpha^M) = t_{\rho(\alpha)}^N$, independently of M and N. Since Dehn twists about nonseparating circles are conjugate in Mod_S, the integers M and N may be chosen independently of $\alpha \in V_0(S)$. For each nonseparating circle a on S, let $\rho(a)$ be a realization of $\rho(\alpha)$, where α is the isotopy class of a. Then $\rho(a)$ is well defined up to isotopy on S' and $\rho(t_a^M) = t_{\rho(a)}^N$. Note that, clearly, $\rho(a)$ is the canonical reduction system of $t_{\rho(a)}^N$ and, since $t_{\rho(a)}^N = \rho(t_a^M) = \rho(t_a)^M$, it is also the canonical reduction system of $\rho(t_a)$.

For the remainder of this section we will denote by g, b (respectively g', b') the genus and the number of the boundary components of S (respectively S').

Lemma 10.1. (i) *$\rho(\alpha) = \rho(\beta)$ if and only if $\alpha = \beta$.*

(ii) *Let C be a system of nonseparating circles on S and σ be the corresponding simplex of $C(S)$. Then $\rho(\sigma) = \{\rho(\alpha) : \alpha \in \sigma\}$ is a simplex of $C(S')$.*

Proof. (i) The "if" clause is trivial. Since ρ is almost twist-preserving, $\rho(\alpha) = \rho(\beta)$ implies $\rho(t_\alpha^M) = \rho(t_\beta^M)$. Since ρ is injective, this, in turn, implies that $t_\alpha^M = t_\beta^M$. Hence, by Theorem 4.1, $\alpha = \beta$.

(ii) Let $\alpha, \beta \in \sigma$. Then t_α and t_β commute. This implies that $\rho(t_\alpha^M)$ and $\rho(t_\beta^M)$ commute or, what is the same, $t_{\rho(\alpha)}^N$ and $t_{\rho(\beta)}^N$ commute. By Theorem 4.2, this implies that $i(\rho(\alpha), \rho(\beta)) = 0$. The assertion (ii) follows. □

Lemma 10.2. *Suppose only that there exists an injective (not necessarily almost twist-preserving) homomorphism* $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$. *Then the following holds:*

(i) *S' is not a sphere with at most three holes;*

(ii) *if S' is a sphere with four holes or a torus with one hole, then S is a torus with one hole;*

(iii) *if S' is a sphere with five holes or a torus with two holes, then S is a torus with one or two holes.*

Proof. Since, by our assumptions, S is of positive genus and not a closed torus, the maxima of ranks of abelian subgroups of Mod_S is equal to $3g-3+b$ (cf. 2.1). Again, since S is of positive genus and not a closed torus, $3g-3+b \geq 1$. On the other hand, if S' is a sphere with at most three holes, then the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is 0. This implies (i). If S' is a sphere with four holes or a torus with one hole, then the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is 1. It follows that $3g-3+b \leq 1$ and $3g+b \leq 4$. Since $g \geq 1$ and $b \geq 1$ if $g = 1$ (because S is not a closed torus), this implies that $g = b = 1$ and S is a torus with one hole. This proves (ii). If S' is a sphere with five holes or a torus with two holes, then the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is 2. It follows that $3g-3+b \leq 2$ and $3g+b \leq 5$. Since $g \geq 1$, this implies $b \leq 2$. Because S is not a closed torus, this proves (iii). □

Lemma 10.3. *Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one (and that $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective almost twist-preserving homomorphism). Then the following holds:*

(i) *if S' is a sphere with six holes, then S is a torus with two or three holes;*

(ii) *if S' is a sphere with seven holes, then S is a torus with three or four holes.*

(iii) *if S' is a sphere with eight holes, then S is a torus with four or five holes.*

Proof. Let C be a maximal system of nonseparating circles on S and σ be the corresponding simplex of $C(S)$. Let $\rho(C)$ be a realization of the simplex $\rho(\sigma) = \{\rho(\alpha) : \alpha \in \sigma\}$ (cf. Lemma 10.1 (ii)).

Since S is of positive genus and not a closed torus, C consists of $3g-3+b$ components and $3g-3+b$ is also the maxima of ranks of abelian subgroups of Mod_S. By Lemma 10.1 (i) $\rho(C)$ also consists of $3g-3+b$ components. Since Dehn twists along nonseparating circles are all conjugate in Mod_S, all elements $t_\alpha^M, \alpha \in \sigma$ are conjugate in Mod_S and, hence, all elements $t_{\rho(\alpha)}^N = \rho(t_\alpha^M), \alpha \in \sigma$ are conjugate in $\mathrm{Mod}_{S'}$. Now, Theorem 4.1 and the fact that $f t_{\rho(\alpha)}^N f^{-1} = t_{f(\rho(\alpha))}^N$ for any $f \in \mathrm{Mod}_{S'}$ imply that all components of $\rho(C)$ are topologically equivalent on S' in the sense of Section 9. Since S' is a sphere with holes in all our cases (i)–(iii), all components of $\rho(C)$ are separating. This fact together with the assumption on the maxima of ranks of abelian subgroups will allow us to apply the results of Section 9. After these preliminary remarks we now proceed with the proofs of the statements (i)–(iii).

(i) If S' is a sphere with six holes, then the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is equal to 3. By our assumption, $3g-3+b \leq 3 \leq 3g-3+b+1$ and $3g+b \leq 6 \leq 3g+b+1$. Since $g \geq 1$, this implies that either $g=1$ and $b=2,3$ or $g=2$ and $b=0$. We have to exclude the last case.

Suppose that $g=2$ and $b=0$, i.e. that S is a closed surface of genus 2. In this case C and $\rho(C)$ consist of 3 components. Note that $3 = 3g'-3+b'$ because S' is a sphere with 6 holes. As we had seen, all components of $\rho(C)$ are topologically equivalent. Hence, we may apply Theorem 9.2 to S' and $\rho(C)$ in the role of S and C respectively. We conclude that for each component a' of $\rho(C)$ there exist a disc with two holes P' embedded in S' such that $\partial P'$ consists of a' and two components of $\partial S'$. Clearly, these discs with two holes are disjoint and the closure of their complement in S' is another disc with holes embedded in S', which we will denote Q'. Obviously, $\partial Q' = \rho(C)$.

Let us consider now the hyperelliptic involution $i \in \mathrm{Mod}_S$ and its image $i' = \rho(i) \in \mathrm{Mod}_{S'}$. Together with i, the image i' is a non-trivial element of finite order (actually of order two). By a well known theorem of Nielsen (cf. for example, [FLP], Exp. 11, §4), it can be realized by a non-trivial periodic diffeomorphism $F' : S' \to S'$. Moreover, we may assume that F' is an isometry of a hyperbolic metric on S' with geodesic boundary. In addition, we may assume that the components of $\rho(C)$ are geodesics with respect to this metric. Now, it is well known that i is the (unique) non-trivial element of the center of Mod_S. In particular, i commutes with the Dehn twists along the components of C. It follows that i' commutes with the N-th powers of the Dehn twists along the components of $\rho(C)$. Now, Theorem 4.1 and the relation $i' t_{\alpha'}^N (i')^{-1} = t_{i'(\alpha')}^N$ (where $\alpha' \in V(S')$) imply that i' preserves the isotopy classes of the components of of $\rho(C)$. Hence, F' preserves the components of $\rho(C)$ themselves (we chose them to be unique geodesic representatives of their isotopy classes). This, clearly, implies that F' preserves the disc with two holes Q'. The diffeomorphism $Q' \to Q'$ induced by F' preserves orientation and preserves each component of the boundary. Hence, it is isotopic to the identity (cf., for example, [FLP], Exp. 2, §III). Being an isometry, it is actually the identity. Hence, F' is equal to the identity on Q'. Because F' is an isometry, this implies that F' is equal to the identity on the whole S' and, hence, $i' = 1$. This contradicts to the injectivity of ρ. Hence, S cannot be a closed surface of genus 2. This completes the proof of (i).

(ii) If S' is a sphere with seven holes, then the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is equal to 4. By our assumption, $3g-3+b \leq 4 \leq 3g-3+b+1$ and $3g+b \leq 7 \leq 3g+b+1$. Since $g \geq 1$, this implies that either $g=1$ and $b=3,4$ or

$g = 2$ and $b = 0, 1$. We have to exclude the case $g = 2$.

Suppose first that $g = 2$ and $b = 1$. In this case C and $\rho(C)$ consist of 4 components. Note that $4 = 3g' - 3 + b'$ because S' is a sphere with 7 holes. As we had seen, all components of $\rho(C)$ are topologically equivalent, and hence, we may apply Theorem 9.2 exactly as in the proof of (i). We conclude that for each component a' of $\rho(C)$ there exist a disc with two holes P' embedded in S' such that $\partial P'$ consists of a' and two components of $\partial S'$. Clearly, these discs with two holes are disjoint. Each of them contributes two components to the boundary $\partial S'$. This implies that $\partial S'$ has at least 8 components. Contradiction with the assumption that S' is a sphere with 7 holes completes our consideration of the $g = 2, b = 1$ case.

Assume now that $g = 2, b = 0$, i.e. that S is a closed surface of genus 2. In this case C and $\rho(C)$ consist of 3 components. Note that $3 = 3g' - 4 + b'$ because S' is a sphere with 7 holes. Since all components of $\rho(C)$ are topologically equivalent, this means that we may apply Theorem 9.1 in this case. Again, we conclude that for each component a' of $\rho(C)$ there exist a disc with two holes P' embedded in S' such that $\partial P'$ consists of a' and two components of $\partial S'$. Clearly, these discs with two holes are disjoint and the closure of their complement in S' is a sphere with four holes embedded in S', which we denote Q'. One component of the boundary $\partial Q'$ is a part of $\partial S'$ and the other components of $\partial Q'$ are components of $\rho(C)$. Arguing exactly as in the proof of (i), we can realize the image $i' = \rho(i)$ of the hyperelliptic involution i by an isometry $F' : S' \to S'$ of a hyperbolic metric on S' with geodesic boundary such that F' preserves all the components of $\rho(C)$ (we assume that they are geodesic). Such an F' obviously preserves Q', and, preserving three of the four components of the boundary $\partial Q'$, it preserves them all. Since F' is orientation-preserving, it follows that F' acts trivially on the first homology group of Q' (with any coefficients). This implies that F' is equal to the identity on Q'. (Note that F' is periodic and use, for example, [I3], Theorem 1.3.) It follows that F' is equal to the identity on the whole surface S' and, hence, $i' = 1$. As in the proof of (i), this contradicts to the injectivity of ρ. Hence, S cannot be a closed surface of genus 2. This completes the proof of (ii).

(iii) If S' is a sphere with eight holes, then the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is equal to 5. By our assumption, $3g - 3 + b \leq 5 \leq 3g - 3 + b + 1$ and $3g + b \leq 8 \leq 3g + b + 1$. Since $g \geq 1$, this implies that either $g = 1$ and $b = 4, 5$ or $g = 2$ and $b = 1, 2$. We have to exclude the case $g = 2$.

Suppose first that $g = 2$ and $b = 2$. In this case C and $\rho(C)$ consist of 5 components. Note that $5 = 3g' - 3 + b'$ because S' is a sphere with 8 holes. Using Theorem 9.2 exactly as in the case $g = 2, b = 1$ of the proof of (ii), we conclude that $\partial S'$ has at least 10 components. The obvious contradiction completes our consideration of the $g = 2, b = 2$ case.

Suppose now that $g = 2$ and $b = 1$, i.e. S is a surface of genus two with one boundary component. In this case C and $\rho(C)$ consist of 4 components and $4 = 3g' - 4 + b'$. So, Theorem 9.1 applies. Hence, for each component a of C there exist a disk with two holes P'_a embedded in S' such that $\partial P'_a$ consists of $\rho(a)$ and two components of ∂S. These discs with two holes are disjoint and the closure of their complement in S is a sphere with two holes, which we denote by Q'.

We may assume that C consists of circles a_1, a_3, b_4, c_4 presented on the Figure 7.1. Clearly, there exists a circle e such that $i(a_1, e) = i(b_4, e) = i(c_4, e) = 1$ and $i(a_3, e) = 0$. Let a be any of the circles a_1, b_4, c_4. By our assumptions, $\rho(t_a)^M = \rho(t_a^M) = t_{\rho(a)}^N$ is a power of a Dehn twist along $\rho(a)$. We will prove now that the element $\rho(t_a)$ itself is a power of the Denh twist along $\rho(a)$.

If b is a component of C, then t_a commutes with t_b and, hence, $\rho(t_a)$ commutes with $\rho(t_b^M) = t_{\rho(b)}^N$. By the usual argument (compare the proof of the fact that i' preserves the isotopy classes of components of $\rho(C)$ in the proof of (i)) this implies that $\rho(t_a)$ preserves the isotopy classes of all components of $\rho(C)$. Hence, we can represent $\rho(t_a)$ by a diffeomorphism $H' : S' \to S'$ preserving all components of $\rho(C)$. Let R' be the result of cutting S' along $\rho(a)$. The surface R' consists of two components, one of them is a disc with two holes P'_a and the other is a sphere with seven holes. We denote the second component by Q''. Clearly, Q'' contains Q'. Since the diffeomorphism H' preserves $\rho(a)$, it induces a diffeomorphism $G' : R' \to R'$. Since $(H')^M$ represents $t_{\rho(a)}^N = \rho(t_a^M)$, $(G')^M$ is isotopic to the identity. Since the two components of R' are not diffeomorphic, G' preserves them both. Let G'' be the diffeomorphism $Q'' \to Q''$ induced by G'. By using the same theorem of Nielsen as in the proof of (i), we can find a hyperbolic metric with geodesic boundary on Q'' and an isometry F' of this metric isotopic to G''. In addition, we may assume that components of $\rho(C) \setminus \rho(a)$ are geodesic with respect to this metric. Since F' together with G'' and H' preserves the isotopy classes of these components, it has to preserve the components themselves (because F' is an isometry). This implies that F' preserves Q' and all its boundary components. By the same token as in the proof of (ii), this implies that F' is equal to the identity on Q' and, hence, on the whole surface Q''. Hence, the restriction G'' of G' to Q'' is isotopic to the identity. If the restriction of G' to P'_a is also isotopic to the identity, then the diffeomorphism H' representing $\rho(t_a)$ is isotopic to a power of the Dehn twist along $\rho(a)$ as claimed.

The restriction of G' to P'_a preserves the boundary component of P'_a corresponding to $\rho(a)$. If G' is not isotopic to the identity, then it transposes two other components (cf. for example, [FLP], Exp. 2, §III). In other words, $\rho(t_a)$ transposes the boundary components of $\partial S'$ contained in $\partial P'_a$ (and fixes other components). Since all t_a for $a = a_1$, b_4, c_4 are conjugate, if this is true for one of them, then it is true for the remaining two. In this case we can label the components of $\partial S'$ by the numbers $1, 2, \ldots, 8$ in such a way that, say, $\rho(t_{a_1})$ induces the transposition (12), $\rho(t_{b_4})$ induces the transposition (34) and $\rho(t_{c_4})$ induces the transposition (56). Since t_e is also conjugate to t_{a_1}, t_{b_4}, t_{c_4} (because $i(a_1, e) = 1$, e is a nonseparating circle), its image $\rho(t_e)$ also induces some transposition (ij). Now, $t_a t_e t_a = t_e t_a t_e$ because $i(a, e) = 1$ for all $a = a_1,\ b_4,\ c_4$. It follows that

$$(12)(ij)(12) = (ij)(12)(ij)$$
$$(34)(ij)(34) = (ij)(34)(ij)$$
$$(56)(ij)(56) = (ij)(56)(ij).$$

Suppose that $\{i, j\}$ and $\{1, 2\}$ are disjoint. Then (ij) and (12) commute. Since $(12)(ij)(12) = (ij)(12)(ij)$, this implies that $\{i, j\} = \{1, 2\}$. Hence $\{1, 2\}$ and $\{i, j\}$ cannot be disjoint. Likewise $\{i, j\}$ and $\{3, 4\}$ are not disjoint and $\{i, j\}$ and $\{5, 6\}$ are not disjoint. But, clearly $\{i, j\}$ cannot intersect three disjoint sets $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ simultaneously. The contradiction shows that $\rho(t_a)$ fixes all boundary components of $\partial S'$ for $a = a_1,\ b_4,\ c_4$. As we had seen, this means that $\rho(t_a)$ is a power of the Dehn twist along $\rho(a)$ for $a = a_1,\ b_4,\ c_4$.

Now, we are going to use the relation $t_a t_e t_a = t_e t_a t_e$ for, say, $a = a_1$ once more. Since t_e is conjugate to t_a, $\rho(t_e)$ is a power of the Dehn twist along $\rho(e)$. The above relation implies $\rho(t_a)\rho(t_e)\rho(t_a) = \rho(t_e)\rho(t_a)\rho(t_e)$. Since ρ is injective, $\rho(t_a)$ and $\rho(t_e)$ do not commute. Hence, $\rho(a) \neq \rho(e)$. Thus, Theorem 4.2 implies that $i(\rho(a),\ \rho(e)) = 1$. But since S' is a sphere with holes, this is impossible (all circles on S' are separating!). The contradiction completes the proof of (iii). □

10.4. Exceptional pairs. The following pairs (S, S') will be called *exceptional pairs*:

(i) S' is a sphere with four holes or a torus with with one hole and S is a torus with one hole;

(ii) S' is a sphere with five holes and S is a torus with one or two holes;

(iii) S' is a sphere with six holes and S is a torus with two or three holes;

(iv) S' is a sphere with seven holes and S is a torus with three or four holes;

(v) S' is a sphere with eight holes and S is a torus with four or five holes;

(vi) S' is a torus with two holes and S is a torus with one or two holes.

This definition is motivated by Lemmas 10.2 and 10.3 and the following results.

Lemma 10.5. *Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. Suppose that* (S, S') *is not an excluded pair. If a is a nonseparating circle on S, then $\rho(a)$ is a nonseparating circle on S.*

Proof. Let C be a maximal system of nonseparating circles on S containing a and σ be the corresponding simplex of $C(S)$. Let $\rho(C)$ be a realization of $\rho(\sigma)$. By Lemma 10.1, $\rho(C)$ is a system of circles on S' with $3g-3+b$ components. Since Dehn twists about nonseparating circles on S are conjugate in Mod_S, the components of $\rho(C)$ are topologically equivalent circles on S'. Hence, the result follows from Theorems 9.1 and 9.2 and Lemmas 10.2 and 10.3. □

Lemma 10.6. *Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. Suppose that* (S, S') *is not an excluded pair. Suppose, in addition, that* S *is not a closed surface of genus 2. Then* $\rho(t_a)$ *is equal to* $t_{\rho(a)}$ *or* $t_{\rho(a)}^{-1}$.

Proof. By Lemma 10.2, S' is not a disc with two holes, a sphere with four or five holes, or a torus with one or two holes. By assumption, S' is not a closed surface of genus 2.

Let C be a maximal system of nonseparating circles containing a and let σ be the corresponding simplex of $C(S)$. Let $\rho(C)$ be a realization of $\rho(\sigma)$. We may assume that the circles $\rho(a)$, where a runs over components of C, are components of $\rho(C)$ (the circles $\rho(a)$ are well defined only to isotopy). By Lemma 10.5, these circles $\rho(a)$ are nonseparating.

By the usual argument (compare the proof of Lemma 10.3) $\rho(t_a)$ preserves all vertices of $\rho(\sigma)$. In particular, $\rho(t_a)$ preserves the isotopy class of $\rho(a)$. Hence, we can represent $\rho(t_a)$ by a diffeomorphism $H' : S' \to S'$ such that $H'(\rho(a)) = \rho(a)$. Let S'' be the surface obtained by cutting S' along $\rho(a)$ and let $G' : S'' \to S''$ be the diffeomorphism induced by H'. Note that S'' is connected (because $\rho(a)$ is nonseparating). Since $\rho(t_a)^M = \rho(t_a^M) = t_{\rho(a)}^N$ is a power of the Dehn twist along $\rho(a)$, the isotopy class of G' has finite order. Using the Nielsen theorem as in the proof of Lemma 10.3 we choose a hyperbolic metric with geodesic boundary on S'' and an isometry $F' : S'' \to S''$ isotopic to G'.

In addition we may assume that $\rho(b)$ is a geodesic on S'' for each component b of $C \setminus a$. Together with H', the diffeomorphism G'preserves the isotopy classes of all components $\rho(b)$ of $\rho(C) \setminus \rho(a)$. Since F' is an isometry isotopic to G', F' preserves the components $\rho(b)$ themselves.

Let R' be the surface obtained by cutting S'' along all circles $\rho(b)$, where b runs over components of $C \setminus a$. Note that at the same time R' is the result of cutting of S' along $\rho(C)$. The number of component of $\rho(C)$ is equal to the maxima of ranks of abelian subgroups of Mod_S, and it differs by at most one from the corresponding maxima for $\mathrm{Mod}_{S'}$. It follows that $\rho(C)$ is either a maximal system of circles on S', or has one circle less than such a maximal system. Hence, all components of R' are discs with two holes or spheres with four holes, and there is at most one sphere with four holes among them. If there is only one component of R' and it is a sphere with four holes, then S' is either a sphere with four holes, a torus with two holes or a closed surface of genus 2. The first two cases are impossible in view of Lemma 10.2, because (S, S') is not an exceptional pair. The last one is also impossible, by the assumption of the Lemma. Hence, at least one component Q' of R' is a disc with two holes. Since each component of C' is a nonseparating circle and S' is not a torus with one hole (by Lemma 10.2, in view of the fact that (S, S') is not an exceptional pair), Q' is embedded in S'.

If only one component of $\partial Q'$ corresponds to a component of $\rho(C)$, then this component of $\rho(C)$ is separating. As we had seen, all components of $\rho(C)$ are nonseparating. Hence, at least two components of $\partial Q'$ correspond to components of $\rho(C)$. Recall that F' preserves all the components of $\rho(C) \setminus \rho(a)$. In particular, F' preserves at least two components of $\partial Q'$, namely, the components corresponding to components of $\rho(C)$ (here we consider Q' as a subsurface of S''). If $F'(Q') \neq Q'$, then $F'(Q') \cup Q'$ is a subsurface of S'' with the boundary contained in the boundary of S''. Clearly, $F'(Q') \cup Q' = S''$ in this case and, hence, S'' is either a sphere with four holes, a torus with two holes or a closed surface of genus two. Since S'' is the result of cutting of S', it cannot be closed. So, the last case is impossible. In the first two cases S is either a torus with two holes or a closed surface of genus 2. As we had already seen, this is impossible. Hence, $F'(Q') = Q'$. Because F' preserves each component of $\partial Q'$, the diffeomorphism $Q' \to Q'$ induced by F' is isotopic to the identity. Since F' is an isometry, this diffeomorphism is, in fact, the identity. So, the restriction of F' on Q' is the identity and, hence, F' is the identity itself.

Hence, $G' : S'' \to S''$ is isotopic to the identity and $H' : S' \to S'$ is isotopic to a power of the Dehn twist along $\rho(a)$. In other words, $\rho(t_a) = t^K_{\rho(a)}$ for some integer K, which has to be nonzero, because ρ is injective. Since a is nonseparating circle on S, we may choose a nonseparating circle e on S such that $i(a,e) = 1$. Since t_a and t_e are conjugate, $\rho(t_e) = t^K_{\rho(e)}$. Now, $t_a t_e t_a = t_e t_a t_e$ and, hence $\rho(t_a)\rho(t_e)\rho(t_a) = \rho(t_e)\rho(t_a)\rho(t_e)$, i.e. $t^K_{\rho(a)} t^K_{\rho(e)} t^K_{\rho(a)} = t^K_{\rho(e)} t^K_{\rho(a)} t^K_{\rho(e)}$. In addition, $t^K_{\rho(a)}$ and $t^K_{\rho(e)}$ do not commute (because ρ is injective and t_a and t_e do not commute) and thus $\rho(a) \neq \rho(e)$. Hence, Theorem 4.2 implies that $K = \pm 1$. (Compare the end of the proof of Lemma 10.3 (iii)). This completes the proof. □

Lemma 10.7. *Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. Suppose that* S' *is a closed surface of genus* 2 *and* S *is not a torus with two holes. Let* i *be the isotopy class of the hyperelliptic involution on* S'*. Then for any nonseparating circle* a *on* S *the image* $\rho(t_a)$ *is equal to* $t_{\rho(a)}$, $t^{-1}_{\rho(a)}$, $t_{\rho(a)} i$, $t^{-1}_{\rho(a)} i$.

Proof. The maxima of ranks of abelian subgroups of Mod_S is $3g-3+b$, and of $\mathrm{Mod}_{S'}$ is 3. By our assumptions, $3g-3+b \leq 3 \leq 3g-3+b+1$ and $3g+b \leq 6 \leq 3g+b+1$. Hence, S is either a torus with two or three holes or a closed surface of genus two. The first case is excluded by our assumptions, so S is either a torus with three holes or a closed surface of genus two. In both cases any maximal system of nonseparating circles on S consists of three circles.

Let C be a maximal system of nonseparating circles on S containing a and let σ be the corresponding simplex of $C(S)$. Let b and c be two other components of C. Let $\rho(C)$ be a realization of the simplex $\rho(\sigma)$. We may assume that $\rho(a)$, $\rho(b)$ and $\rho(c)$ are the three components of $\rho(C)$. The circles $\rho(a)$, $\rho(b)$ and $\rho(c)$ are nonseparating by Lemma 10.5. It follows that $\rho(C)$ separates S' into two discs with two holes, which we denote by P' and Q'. Both of them are embedded in S'.

By our usual argument (compare the proofs of Lemmas 10.3 and 10.6), $\rho(t_a)$ preserves all vertices of $\rho(\sigma)$. Hence, we can represent $\rho(t_a)$ by a diffeomorphism $F' : S' \to S'$ preserving all components of $\rho(C)$. Clearly, either $F'(P') = P'$, $F'(Q') = Q'$ or $F'(P') = Q'$, $F'(Q') = P'$. We would like to reduce our considerations to the first case. To this end, recall the well known fact that i can be represented by a diffeomorphism $I : S' \to S'$ preserving all components of $\rho(C)$ and such that $I(P') = Q'$, $I(Q') = P'$ (compare the proof of Theorem 6.3). Put $H' = F'$ if $F'(P') = P'$ and $H' = I \circ F'$ if $F'(P') = Q'$ and consider H' instead of F'. The diffeomorphism H' preserves both P' and Q' and preserves all components of $\rho(C) = \partial P' = \partial Q'$. Since both P' and Q' are discs with two holes, it follows that diffeomorphisms $P' \to P'$, $Q' \to Q'$ induced by H' are both isotopic to the identity. This implies that H' is isotopic to a product of powers of Dehn twists about components of $\rho(C)$. In other words, $h' = t^u_{\rho(a)} \circ t^v_{\rho(b)} \circ t^w_{\rho(c)}$ for some integers u, v, w, where h' is the isotopy class of H'. By the construction, $h' = \rho(t_a)$ or $h' = i\rho(t_a)$.

Since i is a central element of order 2, it follows that

$$\rho(t_a)^2 = t^{2u}_{\rho(a)} \circ t^{2v}_{\rho(b)} \circ t^{2w}_{\rho(c)}$$

and, hence,

$$t^{2N}_{\rho(a)} = \rho(t_a)^{2M} = t^{2uM}_{\rho(a)} \circ t^{2vM}_{\rho(b)} \circ t^{2wM}_{\rho(c)}.$$

Since the circles $\rho(a)$, $\rho(b)$ and $\rho(c)$ are pairwise disjoint and nonisotopic, this implies that $2vM = 2wM = 0$, i.e. $v = w = 0$. It follows that $\rho(t_a) = t^u_{\rho(a)}$ or $it^u_{\rho(a)}$.

Since a is a nonseparating circle on S, we may choose a nonseparating circle e on S such that $i(e, a) = 1$. Since t_a and t_e are conjugate, $\rho(t_e) = t^u_{\rho(e)}$ if $\rho(t_a) = t^u_{\rho(a)}$ and $\rho(t_e) = it^u_{\rho(e)}$ if $\rho(t_a) = it^u_{\rho(a)}$ (remember that i is a central element). Now, $t_a t_e t_a = t_e t_a t_e$ and hence, $\rho(t_a)\rho(t_e)\rho(t_a) = \rho(t_e)\rho(t_a)\rho(t_e)$. Since i is a central element, the last equality implies that

$$t^u_{\rho(a)} t^u_{\rho(e)} t^u_{\rho(a)} = t^u_{\rho(e)} t^u_{\rho(a)} t^u_{\rho(e)}$$

in all cases. In addition, $\rho(t_a)$ and $\rho(t_e)$ and, hence, $t^u_{\rho(a)}$ and $t^u_{\rho(e)}$ do not commute (use once again that i is a central element). Hence, Theorem 4.2 implies that $u = \pm 1$. This completes the proof. □

Theorem 10.8. *Let S and S' be compact connected orientable surfaces. Suppose that S has genus at least 2 and S' is not a closed surface of genus 2. If the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one and $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective almost twist-preserving homomorphism, then ρ is induced by a diffeomorphism $S \to S'$.*

Proof. Since the genus of S is at least 2, (S, S') is not one of the exceptional pairs listed above. Let a be a nonseparating circle on S. By Lemma 10.6, $\rho(t_a)$ is equal to $t_{\rho(a)}$ or $t_{\rho(a)}^{-1}$.

Suppose that $\rho(t_a) = t_{\rho(a)}$. Since Dehn twists along nonseparating circles are conjugate in S, it follows that $\rho(t_b) = t_{\rho(b)}$ for every nonseparating circle b on S. In other words, ρ is twist-preserving. Hence, if $\rho(t_a) = t_{\rho(a)}$, the result follows from Theorem 8.9.

Suppose that $\rho(t_a) = t_{\rho(a)}^{-1}$. Let $F' : S' \to S'$ be an orientation-reversing diffeomorphism. F' induces an automorphism $F'_* : \mathrm{Mod}_{S'} \to \mathrm{Mod}_{S'}$. Since F' is orientation reversing, $F'(t_{a'}) = t_{F'(a')}^{-1}$ for every circle a' on S'. Let $\rho' = F'_* \circ \rho$. Then $\rho' : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective homomorphism and $\rho'(t_a) = F'_*(t_{\rho(a)}^{-1})) = t_{F'(\rho(a))}$. Hence, by the previous paragraph, ρ' is twist-preserving. By Theorem 8.9, $\rho' = H'_*$ for some diffeomorphism $H' : S \to S'$. Thus, $\rho = (F'_*)^{-1} \circ H'_*$. This implies that $\rho = ((F')^{-1} \circ H')_*$. This completes the proof.

Note that if S is a closed surface of genus at least 2, only the easy part of the Theorem 8.9 is actually need (cf. the first paragraph of the proof of Theorem 8.9). In particular, we don't need the results of Section 5 in this case. □

Theorem 10.9. *Let S and S' be compact connected orientable surfaces. Suppose that S has positive genus and is not a closed torus. If S is a torus with one hole suppose, in addition, that S' is not a closed torus. Further, suppose that S' is not a closed surface of genus 2 and (S, S') is not an exceptional pair. If the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by at most one and $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective almost twist-preserving homomorphism, then ρ is induced by a diffeomorphism $S \to S'$.*

Proof. The proof is similar to the proof of Theorem 10.8, relying upon Theorem 8.15 rather than Theorem 8.9. □

Theorem 10.10. *Let S be a compact connected orientable surface of positive genus Suppose that S is not a closed torus or a torus with two holes. Let S' be a closed surface of genus 2. Let τ be the exceptional automorphism of $\mathrm{Mod}_{S'}$ given by the rule $\tau(t_{a'}) = it_{a'}$, where $i \in \mathrm{Mod}_{S'}$ is the hyperelliptic involution (cf.* [M] *or* [I2]*). If the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by at most one and $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective almost twist-preserving homomorphism, then either ρ or $\tau \circ \rho$ is induced by a diffeomorphism $S \to S'$.*

Proof. Since S' is a closed surface of genus 2, (S, S') is not an exceptional pair. Hence, Lemma 10.7 applies. The proof is similar to the proof of Theorem 10.8, using Lemma 10.7 and Theorem 8.15 instead of Lemma 10.6 and Theorem 8.9. The automorphism τ compensates for the discrepancy between Lemma 10.7 and Lemma 10.6. The details are left to the reader. □

11. Centralizers of mapping classes

In this section, S denotes a compact connected orientable surface. Let Γ be a subgroup of finite index in Mod_S consisting entirely of pure elements. The goal of this section is to describe the center of the centralizer of elements of Γ. The main results are Theorems 11.6 and 11.7. To a big extent, they are contained implicitly in [I2].

For any group G and a A of G, we denote by $C_G(A)$ the centralizer $\{g \in G : ga = ag$ for all $a \in A\}$ of A in G. For $f \in G$ we denote by $C_G(f)$ the centralizer $\{g \in G : gf = fg\}$ of f in G. Finally, we denote by $C(G)$ the center $C_G(G)$ of G. We are intrested mainly in subgroups $C(C_G(f))$ consisting of all elements of G which commute with every element of G commuting with f.

Lemma 11.1. *Let $f \in \Gamma$, let $\sigma = \sigma(f)$ be the cannonical reduction system for f and C be a realization of σ. Then $C_\Gamma(f) \subset \Gamma(C)$.*

Proof. If h commutes with f, then $h(\sigma) = h(\sigma(f)) = \sigma(hfh^{-1}) = \sigma(f) = \sigma$ and, hence, $h \in M(\sigma)$. Hence, $C_\Gamma(f) \subset \Gamma \cap M(\sigma) = \Gamma(C)$. □

Lemma 11.2. *Let C be a system of circles on S and let B be a subgroup of $\Gamma(C)$ (cf. 2.12 for the notations). If $r_C(B)$ is abelian, then B is also abelian.*

Proof. Recall that the kernel T_C of $r_C : M(\sigma) \to \mathrm{Mod}_{S_C}$, where σ is the simplex corresponding to C, is abelian. Hence, the kernel of $r_C|B : B \to \mathrm{Mod}_{S_C}$ is abelian. Since $r_C(B)$ is abelian, this implies that B is solvable. Finally, Theorem 2.11 implies that B is abelian. □

Lemma 11.3. *Let $f, h \in \Gamma$, let $\sigma = \sigma(f)$ be the canonical reduction system for f and C be a realization of σ. Then $h \in C_\Gamma(f)$ if and only if $h \in \Gamma(C)$ and h_Q commutes with f_Q for every component Q of S_C (cf. 2.3, 2.12 for notations).*

Proof. If h commutes with f, then $h \in \Gamma(C)$ by Lemma 11.1. Since h commutes with f, h_Q commutes with f_Q for every component Q of S_C.

Suppose on the other hand, that $h \in \Gamma(C)$ and h_Q commutes with f_Q for every component Q of S_C. Let B be the subgroup of $\Gamma(C)$ generated by f and h. By the assumption, B_Q is abelian for every component Q of S_C. This implies that $r_C(B)$ is abelian. Now, Lemma 11.2 implies that B is abelian. In particular, h commutes with f. This completes the proof. □

Lemma 11.4. *Let $f \in \Gamma$, $\sigma = \sigma(f)$ and C be a realization of σ. Let Q be a trivial component of S_C with respect to f (cf. 2.3). Then $(C(C_\Gamma(f)))_Q$ is trivial.*

Proof. Let $G = C(C_\Gamma(f))$. If $g \in G$, then $g \in \Gamma$ and, in particular, g is a pure element. Since $f \in C_\Gamma(F)$, it commutes with f and, hence, $g \in \Gamma(C)$ by Lemma 11.1. It follows that σ is a reduction system for g. In particular, g_Q is defined and is a pure element of Mod_Q by Theorem 2.9.

Let $\alpha \in V(Q)$ and let a be a circle on Q in the isotopy class α. We may consider a also as a nontrivial circle on S. Since a is, clearly, disjoint from C, the Dehn twist $t_a \in \mathrm{Mod}_S$ belongs to $M(\sigma)$. Since Γ has finite index in Mod_S, some power $h = t_a^n$, $n \neq 0$ belongs to $\Gamma(C)$. Lemma 11.3 implies that $h \in C_\Gamma(f)$. Since $g \in G = C(C_\Gamma(f))$, g commutes with h. Hence, g_Q commutes with h_Q. The element $h_Q \in \mathrm{Mod}_Q$ is a nontrivial power of the Dehn twist about a on Q. Hence, $\sigma(h_Q) = \{\alpha\}$. Since g_Q commutes with h_Q, it must fix α. This implies that g_Q is in the kernel of the action of Mod_Q on $V(Q)$. Now, Lemmas 6.1, 6.2 imply that g_Q has finite order. It follows that $g_Q = 1$ (because g_Q is pure). This completes the proof. □

Lemma 11.5. *Let $f \in \Gamma$, $\sigma = \sigma(f)$ and C be a realization of σ. Let Q be a pseudo-Anosov component of S_C with respect to f (cf. 2.3). Then $(C(C_\Gamma(f)))_Q$ is an infinite cyclic group.*

Proof. Let $G = C(C_\Gamma(f))$. By Lemma 11.1, $G \subset \Gamma(C) \subset M(\sigma)$. Hence, $G_Q \subset \mathrm{Mod}_Q$ is defined. By Theorem 2.9 G_Q consists entirely of pure elements. Since Q is a pseudo-Anosov component, f_Q is a pseudo-Anosov element. Since G is abelian, G_Q is abelian. The lemma follows now from Theorem 2.10. □

Theorem 11.6. *Let $f \in \Gamma$, $\sigma = \sigma(f)$ be the canonical reduction system for f and C be a realization of σ. Let c be the number of components of C and p be the number of pseudo-Anosov components of f_C. Then $C(C_\Gamma(f))$ is a free abelian group of rank $c + p$.*

Proof. Let $G = C(C_\Gamma(f))$. Since $G \subset \Gamma$, G consists entirely of pure elements of Mod_S. Since G is abelian, Theorem 2.11 implies that G is a free abelian group of rank bounded above by $3g - 3 + b$. It remains to determine the rank of G.

By Lemmas 11.4 and 11.5, the torsion free rank of $r_C(G)$ is bounded above by p. Since the kernel of r_C is a free abelian group of rank c, we conclude that the rank of G is bounded above by $c + p$.

Let $f^{\mathbb{Z}}$ be the cyclic group generated by f. For a component Q of S_C let us consider the cyclic group $f_Q^{\mathbb{Z}}$ generated by $f_Q \in \mathrm{Mod}_Q$. Let Φ be the product of groups $f_Q^{\mathbb{Z}}$ over all components Q of S_C. This product naturally lies in Mod_{S_C}; compare 2.12. Clearly, Φ is a free abelian group of rank p.

Let $\Pi = r_C^{-1}(\Phi)$. Because all elements of Π are obviously pure (C is a pure reduction system for appropriate representatives of all of them), Lemma 11.2 implies that Π is abelian. As we will see in a moment, the restriction $r_C \mid \Pi : \Pi \to \Phi$ is surjective. Given this, the exact sequence $0 \to T_C \to \Pi \to \Phi \to 0$ implies that Π is a free abelian subgroup of rank $p + c$.

In order to show that $\Pi \to \phi$ is surjective, let us choose a diffeomorphism $F \in f$ such that (F, C) satisfies condition P. For each component Q of S_C, let us extend F_Q to a diffeomorphism $\overline{F_Q} : S \to S$ by the identity. If $\overline{f_Q} \in \mathrm{Mod}_S$ is the isotopy class of $\overline{F_Q}$, then $r_C(\overline{f_Q})$ has f_Q as the Q-th coordinate and 1 as all other coordinates (we consider $r_C(\overline{f_Q})$ as an element of the product of groups Mod_R over all components R of S_C; cf. 2.12). The surjectivity follows.

Let $h \in C_\Gamma(f)$. By Lemma 11.3, $h \in \Gamma(C)$ and h_Q commutes with f_Q for every component Q of S_C. Let B be the subgroup of $\Gamma(C)$ generated by $\Pi \cap \Gamma$ and h. Clearly, B_Q is abelian for every component Q of S_C and, hence, $r_C(B)$ is abelian. Since the kernel of r_C is an abelian group, B is a solvable subgroup of Γ. Now, Theorem 2.11 implies that B is abelian and, in particular, h commutes with all elements of $\Pi \cap \Gamma$. In other words, $\Pi \cap \Gamma \subset G$.

Since Γ is of finite index in Mod_S, the intersection $\Pi \cap \Gamma$ is a free abelian group of the same rank $p + c$ as Π. It follows that the rank of G is bounded not only above, but also below by $p + c$. This completes the proof. □

Theorem 11.7. *Let $f \in \Gamma$, $\sigma = \sigma(f)$ be the canonical reduction system for f and C be a realization of σ. For any component Q of S_C, let g_Q be its genus and b_Q be the number of boundary components. Let c be the number of components of C, p be the number of pseudo-Anosov components of S_C with respect to f and t be the sum of the numbers $3g_Q - 3 + b_Q$ over the trivial components Q of S_C. Let any abelian subgroup of $C_\Gamma(f)$ is a free abelian of rank $\leq c + r + t$.*

Proof. Let A be an abelian subgroup of $C_\Gamma(f)$. Since $A \subset C_\Gamma(f) \subset \Gamma$, the subgroup A consists entirely of pure elements. Hence, by Theorem 2.11 A is a free abelian group of finite rank. By Lemma 11.3, $A \subset \Gamma(C)$. This allows us to consider subgroups $A_Q \subset \Gamma(C)_Q \subset \mathrm{Mod}_Q$ for each component Q of S_C. All subgroups A_Q are abelian and, in view of 2.12, $r_C(A)$ is naturally contained in the product of the groups A_Q over all components Q of S_C. Hence, the rank of $r_C(A)$ is bounded by the sum of the ranks of groups A_Q. Since the kernel of r_C is a free abelian group of rank c (cf. 2.1, 2.3), the rank of A is bounded by $c + a$, where a is the above sum.

Let a_Q be the rank of A_Q. By Theorem 2.11, $a_Q \leq 3g_Q - 3 + b_Q$. Clearly, it is sufficient to show that, moreover, $a_Q \leq 1$ if Q is a pseudo-Anosov component. Note that $\Gamma(C)_Q$ consists entirely of pure elements by Theorem 2.9. Since $A \subset C_\Gamma(f)$, the group A_Q is contained in the centralizer of f_Q in $\Gamma(C)_Q$. Hence, Theorem 2.10 implies that $a_Q \leq 1$. This completes the proof. □

11.8. Remark. The bound in Theorem 11.7 is exact. We can construct a free abelian subgroup of $C_\Gamma(f)$ as follows. For each trivial component Q of S_C choose a maximal systenm of circles in Q. Let D be the union of C and all these maximal systems. Clearly, D consists of $c + t$ components. Let us choose, as in the proof of Theorem 11.6, a diffeomorhism $F \in f$ such that (F, C) satisfies condition P. For each componet Q of S_C, extend F_Q to a diffeomorphism $S \to S$ by the identity and denote by $\overline{f_Q}$ its isotopy class. The group B generated by the elements $\overline{f_Q}$ and the Dehn twists about components of D is, clearly, a free abelian group of rank $c+p+t$.

Moreover, all elements of B commute with f. It folows that $A = B \cap \Gamma$ is the required subgroup (note that A is of finite index in B and, hence, has the same rank).

12. Injective homomorphisms I

As in Section 10, S and S' denote compact connected *oriented* surfaces. We assume that S has positive genus and is not a torus with at most one hole and that the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by at most one. If S has genus one, we assume, moreover, that maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ are equal and that (S, S') is not an exceptional pair in the sense of 10.4. In this section ρ will be an injective homorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ or $\mathrm{Mod}^*_S \to \mathrm{Mod}^*_{S'}$. The second case is needed only for the proof of Theorem 2 (and will be used only in the proof of Lemma 14.1). The notion of an *almost twist-preserving* homorphism is defined for the extended modular groups (i.e., in the case $\mathrm{Mod}^*_S \to \mathrm{Mod}^*_{S'}$) exactly as for the usual modular groups (cf. Section 10).

Our first goal in this section is to show that the image under ρ of a (sufficiently high) power of a Dehn twist about a nonseparating circle is a multitwist about at most two circles. Cf. Lemma 12.6. After this we study the basic properties of ρ when ρ is not almost twist-preserving. Cf. 12.7 and Lemmas 12.8, 12.9, 12.12 and 12.13. These properties often easily lead to a contradiction. This allows us prove Theorems 1, 4, 5 and 6 of the Introduction. They appear below as Theorems 12.17, 12.14, 12.16 and 12.15 respectively.

For the remainder of this section we will denote by g, b (respectively g', b') the genus and the number of the boundary components of S (respectively S').

Since S has positive genus and is not a closed torus, the maxima of the ranks of abelian subgroups of Mod_S is equal to $3g - 3 + b$ (cf. 2.1). Moreover, since S is, in addition, not a torus with one hole, $3g - 3 + b \geq 2$. Since ρ is injective, this implies that S' is not a sphere with at most four holes and not a torus with at most one hole. In particular, the maxima of the ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is equal to $3g' - 3 + b'$. Our asumptions together with the injectivity of ρ imply

$$3g + b \leq 3g' + b' \leq 3g + b + 1.$$

Let us fix subgroups of finite index Γ, Γ' in Mod_S, $\mathrm{Mod}_{S'}$ respectively such that both Γ and Γ' consist entirely of pure elements and $\rho(\Gamma) \subset \Gamma'$. (It is sufficient to take a subgroup of finite index Γ' in $\mathrm{Mod}_{S'}$ consisting entirely of pure elements and let $\Gamma = \rho^{-1}(\Gamma') \cap \Gamma_0$, where Γ_0 is a subgroup of finite index in Mod_S consisting entirely of pure elements. Cf. 2.2 for the existence of Γ', Γ_0.)

Lemma 12.1. *Let H be a subgroup of a group G and let $A \subset H$. Then*

$$C(C_G(A)) \cap H \subset C(C_H(A)).$$

Proof. We leave the (easy) proof to the reader. $\square$

Lemma 12.2. *Let $G \subset \Gamma$ be a free abelian group of rank $3g-3+b$. If $f \in G$, then*

$$\operatorname{rank} C(C_{\Gamma'}(\rho(f))) \leq \operatorname{rank} C(C_{\Gamma}(f)) + 1.$$

Proof. Let $f' = \rho(f)$. Let B be the subgroup of Γ' generated by $\rho(G)$ and $C(C_{\Gamma'}(f'))$ and let $A = \rho(G) \cap C(C_{\Gamma'}(f'))$. Since $f \in G$ and G is abelian, $\rho(G) \subset C_{\Gamma'}(f')$. This implies that B is abelian. We have

$$\operatorname{rank} \rho(G) + \operatorname{rank} C(C_{\Gamma'}(f')) = \operatorname{rank} A + \operatorname{rank} B.$$

Since ρ is injective, $\operatorname{rank} \rho(G) = 3g-3+b$. Thus

$$3g-3+b+\operatorname{rank} C(C_{\Gamma'}(f')) = \operatorname{rank} A + \operatorname{rank} B.$$

Since $B \subset \operatorname{Mod}_{S'}$, $\operatorname{rank} B \leq 3g'-3+b'$. Hence,

$$3g+b+\operatorname{rank} C(C_{\Gamma'}(f')) \leq \operatorname{rank} A + 3g' + b'.$$

Since $3g'+b' \leq 3g+b+1$, this implies

$$\operatorname{rank} C(C_{\Gamma'}(f')) \leq \operatorname{rank} A + 1.$$

By Lemma 12.1, $C(C_{\Gamma'}(f')) \cap \rho(\Gamma) \subset C(C_{\rho(\Gamma)}(f'))$. It follows that $A \subset C(C_{\rho(\Gamma)}(f'))$. Since ρ is injective, the last group is isomorphic to $C(C_{\Gamma}(f))$. Hence,

$$\operatorname{rank} A \leq \operatorname{rank} C(C_{\Gamma}(f)).$$

The lemma follows. $\square$

Corollary 12.3. *Let $f \in \Gamma$ be a power of a Dehn twist. Then*

$$\operatorname{rank} C(C_{\Gamma'}(\rho(f))) \leq 2.$$

Proof. If f is a power of a Dehn twist about a nontrivial circle A, then A is a realization of the canonical reduction system and $r_a(f) = 1$. Hence, by Theorem 11.6, $\operatorname{rank} C(C_{\Gamma}(f)) = 1$. It remains to apply Lemma 12.2. $\square$

Lemma 12.4. *Let $f \in \Gamma$ be a power of a Dehn twist. Then $\rho(f)$ is reducible of infinite order.*

Proof. Since f is of infinite order and ρ is injective, $\rho(f)$ is of infinite order. Hence, $\rho(f)$ is either reducible or pseudo-Anosov.

If $\rho(f)$ is pseudo-Anosov, then $C_{\Gamma'}(\rho(f))$ is an infinite cyclic group by Theorem 12.4. Let $f = t_a^n$ for some $n \in \mathbb{Z}$ and some circle a. Let C be a maximal system of circles containing a. Recall that T_C is the subgroup of Mod_S generated by the Dehn twists about components of C. Thus T_C is a free abelian group of rank $3g-3+b$ containing f. It follows that $\rho(T_C \cap \Gamma)$ is also free abelian of rank $3g-3+b$ and $\rho(T_C \cap \Gamma) \subset C_{\Gamma'}(\rho(f))$. Since $C_{\Gamma'}(\rho(f))$ is infinite cyclic, this implies that $3g-3+b \leq 1$ and $3g+b \leq 4$. Since $g \geq 1$ and S is not a torus with at most one hole by the assumptions of this section, this is impossible. Hence, $\rho(f)$ is reducible. $\square$

Lemma 12.5. *If a, b are disjoint nonseparating nonisotopic circles on S, then there exists a nonseparating circle d on S such that $i(d, a) = 0$ and $i(d, b) \neq 0$. Similarly, if a, b, c are disjoint nonseparating circles on S, then there exists a nonseparating circle d on S such that $i(d, a) = i(d, b) = 0$ and $i(d, c) \neq 0$.*

Proof. We will prove only the first assertion, the proof of the second one being completely similar. Clearly, there exists a possibly separating circle e on S such that $i(e, a) = 0$ and $i(e, b) \neq 0$. By a special case of Proposition 1 from Exposé 4, Appendice of [FLP] we have $i(t_e(b), b) = i(e, b)^2$. (Compare the proof of Theorem 4.2.) It follows that $c = t_e(b)$ is the required nonseparating circle. □

Lemma 12.6. *Let $f \in \Gamma$ be a power of a Dehn twist about a nonseparating circle a. Let C' be a realization of the canonical reduction system for $\rho(f)$. Then C' has at most two components and $\rho(f)$ is a multitwist about C' (i.e., an element of $T_{C'}$).*

Proof. Let $f' = \rho(f)$. By Theorem 11.6, $C(C_{\Gamma'}(f'))$ is a free abelian group of rank $c' + p'$, where c' is the number of components of C' and p' is the number of pseudo-Anosov components of $\rho_{C'}(f')$. By Corollary 12.3, $c' + p' \leq 2$. Hence, $c' \leq 2$. This proves the first assertion.

Lemma 12.4 implies that C' is nonempty. Suppose that $p' \neq 0$. Then $c' = 1$ and $p' = 1$. Hence, C' is a nontrivial circle on S' and there is exactly one component P of $S'_{C'}$ such that f'_P is pseudo-Anosov.

Suppose that P is the only component of $S_{C'}$. Then $g \geq 1$. As in the proof of Lemma 12.4, let us consider a maximal system of circles C containing a. Then T_C is a free abelian group of rank $3g - 3 + b$ containing f. It follows that $\rho(T_C \cap \Gamma)$ is also free abelian of rank $3g - 3 + b$ and $\rho(T_C \cap \Gamma) \subset C_{\Gamma'}(\rho(f))$. By Theorem 11.7, rank $\rho(T_C \cap \Gamma) \leq 2$ and, hence, $3g - 3 + b \leq 5$. Since $g \geq 1$ and S is not a torus with at most one hole, we conclude that S is a torus with two holes and, in particular, $g = 1$. Hence, by the assumptions of this section, the maxima of ranks of abelian subgroups of Mod_S and $\text{Mod}_{S'}$ are equal, i.e. $3g - 3 + b = 3g' - 3 + b'$. This, together with $g' \geq 1$ implies that S' is also a torus with two holes. But then (S, S') is an exceptional pair. The contradiction with our assumptions shows that P cannot be the only component of $S'_{C'}$.

Thus, C' is a nontrivial separating circle on S'. Hence, $S'_{C'}$ has exactly two components, P and the other component which we denote Q. Moreover, f'_P is pseudo-Anosov and f'_Q is trivial.

Now, let C be a maximal system of nonsepatrating circles on S containing a. For each component b of C, choose a power $f_b \in \Gamma$ of the Dehn twist about b and let $f'_b = \rho(f_b)$. We may assume that f_a is a power of f. Since Dehn twists about nonseparating circles on S are all conjugate in Mod_S we may assume that all elements f_b are conjugate in Mod_S. It follows that the images f'_b are all conjugate in Mod^*_S and, moreover, are all conjugate to a power of f'. Hence, for each component b of C the canonical reduction system $\sigma(f')$ can be realized by a nontrivial separating circle

$\rho(b)$ on S' dividing S' into two parts P_b and Q_b such that $(f_b')_{P_b}$ is pseudo-Anosov and $(f_b')_{Q_b}$ is trivial. (Clearly, $(\rho(a), P_a, Q_a) = (C', P, Q)$ and the triples $(\rho(b), P_b, Q_b)$ are all topologically equivalent.)

The elements f_b' generate a free abelian group F_C' of rank $3g-3+b$. It follows that the circles $\rho(b)$ can be choosen to be pairwise disjoint or equal and then the union $\rho(C)$ of these circles $\rho(b)$ is a realization of a reduction system for F_C'.

Let m be the number of components of $\rho(C)$. Let $R_1', \ldots, R_n'$ be the components of $S'_{\rho(C)}$ not diffeomorphic to a disc with two holes. Every component R_i', $1 \leq i \leq n$ contains a nontrivial circle. Pick up such a circle in each R_i', $1 \leq i \leq n$ and consider the union C' of $\rho(C)$ and all these circles. Then C' is a system of $m+n$ circles on S'. In particular, $m + n \leq 3g' - 3 + b'$.

If $m, n \geq 3g-3+b$ then $m+n \geq 6g-6+2b$ and, hence, $6g-6+2b \leq 3g'-3+b'$. Since $3g'-3+b' \leq 3g-3+b+1$, this implies that $6g-6+2b \leq 3g-3+b+1$ and $3g+b \leq 4$. Since $g \geq 1$, this means that S is a torus with at most one hole, which is impossible by the assumptions of this section.

Hence, either $m < 3g-3+b$, or $n < 3g-3+b$. If $m < 3g-3+b$, then there exists a pair of distinct components b, d of C such that $\rho(b) = \rho(d)$. Suppose that $n < 3g-3+b$. Note that for any component b of C the componet P_b of $S'_{\rho(b)}$ must be simultaneously a component of $S'_{\rho(C)}$, because $\rho(C)$ is a realization of a reduction system for $f_b' \in F_C'$ and $(f_b')_{P_b}$ is pseudo-Anosov. Moreover, P_b contains a nontrivial circle (because it carries a pseudo-Anosov element) and, hence, P_b is one of the components R_i'. It follows that $P_b = P_d$ for a pair of distinct components b, d of C. Clearly, $\rho(b) = \rho(d)$ in this case. Hence, $\rho(b) = \rho(d)$ for a pair of distinct componets b, d in all the cases.

Let b, d be a pair of distinct components of C such that $\rho(b) = \rho(d)$. Then $\{P_b, Q_b\} = \{P_d, Q_d\}$. Hence, $P_b = P_d$ or Q_d.

Suppose first that $P_d = P_b$. Then $Q_d = Q_b$. By Lemma 12.5, we may choose a third nonseparating circle e on S such that $i(e,b) = 0 \neq i(e,d)$. Choose a power $f_e \in \Gamma$ of the Dehn twist about e and let $f_e' = \rho(f_e)$. Then f_d and f_e commute with f_b but f_e does not commute with f_d. This implies that f_d' and f_e' commute with f_b' and f_e' does not commute with f_d' (the last is because ρ is injective). Let B be the subgroup of Γ' generated by f_b', f_d' and f_e'. Since the generators of B all commute with f_b', they all preserve the isotopy class of $\rho(b)$. Hence, we have a reduction homomorphism $r_{\rho(b)} : B \to \mathrm{Mod}_{R'}$, where $R' = S'_{\rho(b)}$. Since B_{P_b} contains the pseudo-Anosov element $(f_b')_{P_b}$ and every element of B_{P_b} commutes with this element, Theorem 2.10 implies that B_{P_b} is infinite cyclic. On the other hand, since $Q_b = Q_d$ is a trivial component of both $r_{\rho(b)}(f_b')$ and $r_{\rho(d)}(f_d')$, the group B_{Q_b} is generated by $(f_e')_{Q_b}$. Hence, $B_{Q'}$ is abelian for both components Q' of R'. This implies that $r_{\rho(b)}(B)$ is abelian. Now, Lemma 11.2 implies that B is abelian. In particular, f_e' commutes with f_d', in a contradiction with the above.

Hence, $P_d = Q_b$. Thus, the two components of R' are P_b and P_d. Let $A = \{f_b, f_d\}$ and $A' = \rho(A) = \{f'_b, f'_d\}$. Each element of the centralizer $G = C_{\Gamma'}(A')$ preserves the isotopy class of the circle $\rho(b)$. Hence, we have a reduction homomorphism $\rho_{\rho(b)} : G \to \mathrm{Mod}_{R'}$. Since G_{P_b} contains the pseudo-Anosov element $(f'_b)_{P_b}$ and every element of G_{P_b} commutes with this element, Theorem 2.10 implies that G_{P_b} is infinite cyclic. Likewise, G_{P_d} is infinite cyclic. Hence, $G_{Q'}$ is abelian for every component Q' of R' and, hence, $r_{\rho(b)}(G)$ is abelain. Hence, by Lemma 11.2, G is abelian. Because ρ maps $C_\Gamma(A)$ injectively into G this implies that $C_\Gamma(A)$ is abelian.

If b and d are the only components of C, then $3g - 3 + b = 2$ and $3g + b = 5$. Since $g \geq 1$, S is a torus with two holes in this case. By the assumptions of this section, this implies that $3g' - 3 + b' = 2$ and $3g' + b' = 5$. Hence, S' is either a sphere with five holes or a torus with two holes. In both cases (S, S') is an exceptional pair, which is impossible by our assumptions. Hence, there exists a third component e of C. Since b, d and e are disjoint nontrivial circles on S, we may choose a nontrivial circle h on S such that $i(h, b) = i(h, d) = 0 \neq i(h, e)$. Then f_e and f_h are noncommuting elements of $C_\Gamma(A)$. This contradicts to the previous paragraph. The contradiction shows that our assumption $p' \neq 0$ is not true and, hence, proves the lemma. □

12.7. Action of ρ on (the isotopy classes of) circles. Let $\alpha \in V_0(S)$ be the isotopy class of some nonseparating circle a. Let us choose some $n \neq 0$ such that $t_\alpha^n \in \Gamma$ and put $\rho(\alpha) = \sigma(\rho(t_\alpha^n))$. In fact, $\rho(\alpha) = \sigma(\rho(t_\alpha^n)) = \sigma(\rho(t_\alpha)^n) = \sigma(\rho(t_\alpha))$ by the definition of the canonical reduction systems (cf. 2.3). In particular, $\rho(\alpha)$ does not depend on the choice of $n \neq 0$. Let $\rho(a)$ be a realization of $\rho(\alpha)$. By Lemma 12.6, $\rho(a)$ consists of one or two components and $\rho(t_\alpha^n)$ is a multitwist about $\rho(a)$ (or, what is the same, about $\rho(\alpha)$) if $t_\alpha^n \in \Gamma, n \neq 0$.

For $\sigma \subset V_0(S)$ we define $\rho(\sigma)$ as the union of simplices $\rho(\alpha)$ over $\alpha \in \sigma$, in a slight disagreement with the usual set-theoretic notation. As we will see in a moment (cf. Lemma 12.8), if $\sigma \subset V_0(S)$ is a simplex of $C(S)$, then $\rho(\sigma)$ is a simplex of $C(S')$. If C is a system of nonseparating circles on S, then we will denote by $\rho(C)$ a realization of the simplex $\rho(\sigma)$, where σ is the simplex of $C(S)$ corresponding to C. The system of circles $\rho(C)$ is well defined up to isotopy on S'.

Lemma 12.8. *Let $\alpha, \beta \in V_0(S)$. Then $i(\alpha, \beta) = 0$ if and only if $i(\rho(\alpha), \rho(\beta)) = 0$. If $\sigma \subset V_0(S)$ is a simplex, then $\rho(\sigma)$ is a simplex.*

Proof. Clearly, the second assertion follows from the first one. In order to prove the first assertion, let us choose $m, n \neq 0$ such that $t_\alpha^n, t_\beta^m \in \Gamma$. Let $f_\alpha = t_\alpha^n, f_\beta = t_\beta^m$.

If $i(\alpha, \beta) = 0$, then f_α commutes with f_β by Theorem 4.2 and, hence,

$$\rho(f_\beta)(\rho(\alpha)) = \rho(f_\beta)(\sigma(\rho(f_\alpha))) = \sigma(\rho(f_\beta)\rho(f_\alpha)\rho(f_\beta)^{-1}) =$$

$$= \sigma(\rho(f_\beta f_\alpha f_\beta^{-1})) = \sigma(\rho(f_\alpha)) = \rho(\alpha).$$

By combining this with Theorem 2.7 we see that $\rho(f_\beta)$ fixes all vertices of $\rho(\alpha)$. For such a vertex $\gamma \in \rho(\alpha)$ let us choose $k \neq 0$ such that $t_\gamma^k \in \Gamma'$. Since $\rho(f_\beta)(\gamma) = \gamma$, we have

$$t^k_{\rho(f_\beta)(\gamma)} = t^k_\gamma$$

and $\rho(f_\beta) t_\gamma^k \rho(f_\beta)^{-1} = t_\gamma^k$. In other words, t_γ^k commutes with $\rho(f_\beta)$. Arguing as above, we can deduce from this that t_γ^k commutes with some nontrivial power $t_\delta^l \in \Gamma', l \neq 0$ of t_δ for any $\delta \in \rho(\beta)$. Now, Theorem 4.2 implies that $i(\gamma, \delta) = 0$ for any $\gamma \in \rho(\alpha), \delta \in \rho(\beta)$. Hence, $i(\rho(\alpha), \rho(\beta)) = 0$. Conversely, if $i(\rho(\alpha), \rho(\beta)) = 0$, then $\rho(f_\alpha)$ and $\rho(f_\beta)$ commute because they are multitwists about $\rho(\alpha)$ and $\rho(\beta)$ respectively, in view of Lemma 12.6. Since ρ is injective, in this case f_α and f_β commute. Hence, $i(\alpha, \beta) = 0$ by Theorem 4.2. □

Lemma 12.9. *If ρ is not almost twist-preserving, then $\rho(\alpha)$ is an edge of $C(S')$ for any $\alpha \in V_0(S)$.*

Proof. If $\alpha, \beta \in V_0(S)$, then t_α, t_β are conjugate in Mod_S. Hence, for appropriate $n \neq 0$, the powers t_α^n, t_β^n are both in Γ and are conjugate in Mod_S. Then $\rho(t_\alpha^n), \rho(t_\beta^n)$ are conjugate in $\mathrm{Mod}^*_{S'}$ and, hence, $\rho(\alpha)$ and $\rho(\beta)$ are equivalent under the action of $\mathrm{Mod}^*_{S'}$. It follows that either all $\rho(\alpha)$, $\alpha \in V_0(S)$ consist of one vertex, and in this case ρ is almost twist-preserving, or all $\rho(\alpha)$, $\alpha \in V_0(S)$ consist of two vertices, i.e. all $\rho(\alpha)$ are edges of $C(S')$. □

Lemma 12.10. *Suppose that ρ is not almost twist-preserving. Let $\alpha, \beta \in V_0(S)$. If $i(\alpha, \beta) = 0$ and $\alpha \neq \beta$, then $\rho(\alpha) \cup \rho(\beta)$ as a triangle of $C(S')$. In particular, $\rho(\alpha)$ and $\rho(\beta)$ have a unique common vertex.*

Proof. Clearly, $\{\alpha, \beta\}$ is a simplex of $C(S)$. In view of Lemma 12.8, this implies that $\rho(\alpha) \cup \rho(\beta) = \rho(\{\alpha, \beta\})$ is a simplex of $C(S')$.

Suppose that $\rho(\alpha) = \rho(\beta)$. Let C' be a realization of $\rho(\alpha)$ and let $R' = S'_{C'}$. In view of Lemma 12.5 we may choose a vertex $\delta \in V_0(S)$ such that $i(\delta, \alpha) = 0$ and $i(\delta, \beta) \neq 0$. Let f_α, f_β and $f_\delta \in \Gamma$ be some nontrivial powers of Dehn twists about α, β and δ respectively. Then f_β and f_δ commute with f_α but f_δ does not commute with f_β. Let $f'_\alpha = \rho(f_\alpha)$, $f'_\beta = \rho(f_\beta)$ and $f'_\delta = \rho(f_\delta)$. Clearly, f'_β and f'_δ commute with f'_α but f'_δ does not commute with f'_β (the last is because ρ is injective). Let G be the subgroup of Γ' generated by f'_α, f'_β and f'_δ. Since the generators of G all commute with f'_α, they all preserve $\rho(\alpha) = \sigma(\rho(f_\alpha))$.

Hence, $G \subset M(\rho(\alpha)) \cap \Gamma' = \Gamma'(C')$ (cf. 2.12 for the notations) and we can consider the reduction homomorphism $r_{C'}|G : G \to \mathrm{Mod}_{R'}$. Since f'_α and f'_β are multitwists about $\rho(\alpha) = \rho(\beta)$, the reductions $r_{C'}(f'_\alpha)$ and $r_{C'}(f'_\beta)$ are both trivial. Thus $r_{C'}(G)$ is generated by $r_{C'}(f'_\delta)$. Hence, $r_{C'}(G)$ is cyclic and, in particular, abelian. By Lemma 11.2, G is abelian. In particular, f'_δ commutes with f'_β in a contradiction with the above. Hence, $\rho(\alpha) \neq \rho(\beta)$.

Suppose now that $\rho(\alpha)\cap\rho(\beta)=\emptyset$. Again, let $f_\alpha, f_\beta\in\Gamma$ be some nontrivial powers of Dehn twists about α,β respectively. Let $h=f_\alpha f_\beta$ and $h'=\rho(h)$. Note that $h\in T_C\cap\Gamma$, where C is some maximal system of circles such that the corresponding simplex contains α and β. The group $T_C\cap\Gamma$ is a free abelian group of rank $3g-3+b$, because T_C is such a group and Γ is of finite index in Mod_S. Hence, Lemma 12.2 implies that

$$\text{rank}\, C(C'_\Gamma(h'))\leq \text{rank}\, C(C_\Gamma(h))+1. \tag{12.1}$$

Clearly, $\{\alpha,\beta\}$ is the canonical reduction system for h and h is a multitwist about $\{\alpha,\beta\}$. Hence, Theorem 11.6 implies that $\text{rank}\, C(C_\Gamma(h))=2$.

Another application of Theorem 11.6 shows that rank $C(C'_\Gamma(h'))=4$ (note that $\sigma(h')=\rho(\alpha)\cup\rho(\beta)$). Contradiction with (12.1) shows that the intersection $\rho(\alpha)\cap\rho(\beta)$ cannot be empty. Thus, the edges $\rho(\alpha)$ and $\rho(\beta)$ are not disjoint and not equal. This means that the simplex $\rho(\alpha)\cup\rho(\beta)$ has exactly three vertices and the edges $\rho(\alpha)$ and $\rho(\beta)$ have exactly one common vertex. □

Lemma 12.11. *Suppose that ρ is not almost twist-preserving. Let $\{\alpha,\beta,\gamma\}\subset V_0(S)$ be a simplex of $C(S)$. Then $\rho(\alpha)\cap\rho(\beta)=\rho(\beta)\cap\rho(\gamma)=\rho(\gamma)\cap\rho(\alpha)$.*

Proof. Of course, it is sufficient to prove that $\rho(\alpha)\cap\rho(\beta)=\rho(\alpha)\cap\rho(\gamma)$. Note that by Lemma 12.9 $\rho(\alpha)$, $\rho(\beta)$ and $\rho(\gamma)$ are edges of $C(S')$. By Lemma 12.10 each pair of these edges has exactly one common vertex. Let $\{\alpha'\}=\rho(\beta)\cap\rho(\gamma)$, $\{\beta'\}=\rho(\alpha)\cap\rho(\gamma)$ and $\{\gamma'\}=\rho(\alpha)\cap\rho(\beta)$.

Suppose that $\beta'\neq\gamma'$. Then $\rho(\alpha)=\{\beta',\gamma'\}$, because $\beta',\gamma'\in\rho(\alpha)$. Moreover, $\alpha'\neq\beta'$ in this case, because otherwise $\beta'=\alpha'\in\rho(\beta)$ and $\gamma'\in\rho(\beta)$ and, hence, $\rho(\alpha)$ and $\rho(\beta)$ have two common vertices β',γ' in a contradiction with Lemma 12.10. Similarly, $\alpha'\neq\gamma'$ in this case. It follows that if $\beta'\neq\gamma'$, then $\rho(\alpha)=\{\beta',\gamma'\}$, $\rho(\beta)=\{\alpha',\gamma'\}$ and $\rho(\gamma)=\{\alpha',\beta'\}$.

Lemma 12.5 implies that there exists $\delta\in V_0(S)$ such that $i(\delta,\alpha)=i(\delta,\beta)=0$ and $i(\delta,\gamma)\neq 0$. As usual, let $f_\alpha,f_\beta,f_\gamma$ and $f_\delta\in\Gamma$ be some nontrivial powers of Dehn twists about α,β,γ and δ respectively. Since $i(\delta,\alpha)=0$, the elements f_δ and f_α commute. It follows that $\rho(f_\delta)$ and $\rho(f_\alpha)$ commute and, and, since $\rho(\alpha)=\sigma(\rho(f_\alpha))$, that $\rho(f_\delta)(\rho(\alpha))=\rho(\alpha)$. Because $\rho(\alpha)$ consists of only two vertices, this implies that $\rho(f_\delta^2)=\rho(f_\delta)^2$ fixes both vertices of $\rho(\alpha)$. In particular, $\rho(f_\delta^2)(\beta')=\beta'$. Similarly, $i(\delta,\beta)=0$ implies that $\rho(f_\delta^2)(\alpha')=\alpha'$. since $\rho(f_\gamma)$ is a multitwist about $\rho(\gamma)=\{\alpha',\beta'\}$ in view of Lemma 12.6, it follows that $\rho(f_\delta^2)$ commutes with $\rho(f_\gamma)$. Since ρ is injective, this implies that f_δ^2 commutes with f_γ. By Theorem 4.2 this implies that $i(\delta,\gamma)=0$ in a contradiction with the above.

Hence, our assumption that $\beta'\neq\gamma'$ is not true. In other words, $\beta'=\gamma'$ and $\rho(\alpha)\cap\rho(\gamma)=\rho(\alpha)\cap\rho(\beta)$. This completes the proof. □

Lemma 12.12. *Suppose that ρ is not almost twist-preserving. Let σ be a simplex of $C(S)$ contained in $V_0(S)$ and having at least two vertices. Then there exists a unique* *isotopy class* $\rho_\sigma \in V(S')$ *such that* $\rho_\sigma \in \rho(\alpha)$ *for each* $\alpha \in \sigma$. *If* $\alpha, \beta \in \sigma$ *and* $\alpha \neq \beta$, *then* $\{\rho_\sigma\} = \rho(\alpha) \cap \rho(\beta)$.

Proof. Let $\alpha, \beta, \gamma, \delta \in \sigma$ and $\alpha \neq \beta$, $\gamma \neq \delta$. If $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ have a common element, then $\rho(\alpha) \cap \rho(\beta) = \rho(\gamma) \cap \rho(\delta)$ by Lemma 12.11. Otherwise, $\rho(\alpha) \cap \rho(\beta) = \rho(\alpha) \cap \rho(\gamma) = \rho(\gamma) \cap \rho(\delta)$, again by Lemma 12.11. In addition, for any $\alpha, \beta \in \sigma$, $\alpha \neq \beta$ the intersection $\rho(\alpha) \cap \rho(\beta)$ consists of exactly one vertex, by Lemma 12.10. So, we can take ρ_σ to be this vertex. □

Lemma 12.13. *If* ρ *is not almost twist-preserving, then* $3g' + b' = 3g + b + 1$. *Hence, the maxima of ranks of abelian subgroups of* $\mathrm{Mod}_{S'}$ *is bigger by one than the maxima of ranks of abelian groups of* Mod_S.

Proof. Let C be a maximal system of nonseparating circles on S and σ be the corresponding simplex. Let $\rho(C)$ be a realization of $\rho(\sigma)$. By Lemma 12.9, all $\rho(\alpha), \alpha \in \sigma$ are edges. By Lemma 12.12 there is one vertex common to all these edges, and the remaining vertices of these edges are all distinct. This implies that the union $\rho(\sigma)$ of these edges has one more vertex than σ. Hence, $\rho(C)$ has $3g-3+b+1 = 3g-2+b$ components. Since $\rho(C)$ is a system of circles on S', this implies that $3g-2+b \leq 3g'-3+b'$ and $3g + b + 1 \leq 3g' + b'$. On the other hand, as we noticed in the beginning of this section, $3g' + b' \leq 3g + b + 1$. The lemma follows. □

Theorem 12.14. *Let* S *and* S' *be compact connected orientable surfaces. Suppose that* S *has positive genus,* S *is not a torus with at most one hole,* S' *is not a closed surface of genus* 2 *and* (S, S') *is not an exceptional pair. If the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *are equal and* $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ *is an injective homomorphism, then* ρ *is induced by a diffeomorphism* $S \to S'$.

Proof. By Lemma 12.13, ρ is almost twist-preserving. Hence, the result follows from Theorem 10.9. □

Theorem 12.15. *Let* S *be a compact connected orientable surface of positive genus. Let* S' *be a closed surface of genus* 2. *Let* τ *be the exceptional automorphism of* $\mathrm{Mod}_{S'}$. *If the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *are equal and* $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ *is an injective homomorphism, then either* ρ *or* $\tau \circ \rho$ *is induced by a diffeomorphism* $S \to S'$.

Proof. Since S' is a closed surface of genus two, (S, S') is not an exceptional pair. Since $3g - 3 + b = 3g' - 3 + b' = 3$ by the assumption, S is not a torus with at most two holes. By Lemma 12.13, ρ is almost twist-preserving. The result follows now from Theorem 10.10. □

Theorem 12.16. *Let* S *be a compact connected orientable surface of genus at least* 2. *Let* S' *be a closed surface of genus* 2. *Let* τ *be the exceptional outer automorphism* *of* $\mathrm{Mod}_{S'}$. *If* $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ *is an injective homomorphism, then either* ρ *or* $\tau \circ \rho$ *is induced by a diffeomorphism* $S \to S'$.

Proof. Since $g \geq 2$, we have $3g + b \geq 6$. On the other hand, $3g' + b' = 6$ and $3g + b \leq 3g' + b'$, as we noticed in the beginning of this section. Hence, $g = 2$ and $b = 0$, i.e., S is a closed surface of genus two. Now the result follows from Theorem 12.15. □

Theorem 12.17. *Let* S *be a compact connected orientable surface of positive genus. Suppose that* S *is not a torus with at most two holes. Then* Mod_S *is co-Hopfian.*

Proof. Let $S' = S$. The assumptions on S imply that (S, S') is not an exceptional pair. Hence, if S is not a closed surface of genus two, the result follows from Theorem 12.14. Otherwise, the result follows from Theorem 12.15. This completes the proof. □

13. Injective homomorphisms II

As in Section 12, S and S' denote compact connected *oriented* surfaces. In this section we assume that the genus of S is at least two, S' is not a closed surface of genus two and that the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by at most one. As in Section 12, we will denote by g, b (respectively g', b') the genus and the number of the boundary components of S (respectively S'). As usual, let $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ be an injective homomorphism.

The goal of this section is to prove Theorem 3 of the Introduction. It appears below as Theorem 13.7. In addition, we prove a nonsplitting result for modular groups, which follows easily from Theorem 13.7. Cf. Theorem 13.8. We will continue the line of arguments started in Section 12. Note that since S is of genus at least two (in particular, (S, S') is not an exceptional pair), all results of Section 12 are valid under our current assumptions. In particular, we may use the notations introduced in 12.7. As we saw in Section 12, if ρ is not almost twist-preserving, then $\rho(\alpha)$ is an edge of $C(S')$ for any $\alpha \in V_0(S)$ (cf. Lemma 12.9). Moreover, if σ is a simplex of $C(S)$ contained in $V_0(S)$ and having at least two vertices, then there is one vertex common to all edges $\rho(\alpha)$, $\alpha \in \sigma$, and the other vertices of these edges are all distinct (cf. Lemma 12.12). As in Lemma 12.12, we will denote this unique common vertex by ρ_σ.

Lemma 13.1. *Suppose that* ρ *is not almost twist-preserving. Let* C *be the maximal system of circles introduced in* 7.1, *and let* σ *be the simplex of* $C(S)$ *corresponding to* C. *Then* $\rho(\mathrm{PMod}_S)$ *is contained in the stabilizer of* ρ_σ *in* $\mathrm{Mod}_{S'}$.

Proof. All components of C are obviously nonseparating. Hence, $\sigma \subset V_0(S)$. Also, C has $3g - 3 + b$ components and $g \geq 2$. Hence, σ has at least three vertices. In particular, ρ_σ is indeed well defined.

Recall that $\rho(\alpha) = \sigma(\rho(t_\alpha))$ for any $\alpha \in V_0(S)$ (cf. 12.7). It follows that $\rho(\alpha)$ is invariant under $\rho(t_\alpha)$. In addition, if $i(\alpha, \beta) = 0$, then $t_\alpha t_\beta t_\alpha^{-1} = t_\beta$ and $\rho(t_\alpha)(\rho(\beta)) =$

$\rho(t_\alpha)(\sigma(\rho(t_\beta))) = \sigma(\rho(t_\alpha)\rho(t_\beta)\rho(t_\alpha)^{-1}) = \sigma(\rho(t_\alpha t_\beta t_\alpha^{-1})) = \sigma(\rho(t_\beta)) = \rho(\beta)$. So, if $i(\alpha,\beta)=0$, then $\rho(\beta)$ is also invariant under $\rho(t_\alpha)$.

If we apply these remarks to two vertices $\alpha,\beta \in \sigma$, we conclude $\rho(t_\alpha)$ preserves both $\rho(\alpha)$ and $\rho(\beta)$. Since ρ_σ is the unique common vertex of $\rho(\alpha)$ and $\rho(\beta)$ by Lemma 12.12, it follows that ρ_σ is preserved by $\rho(t_\alpha)$. Thus ρ_σ is preserved by all $\rho(t_\alpha)$, $\alpha \in \sigma$.

Now, let β be the isotopy class of one of the *dual* circles of the configuration $\mathcal{C}$; cf. 7.1. We would like to prove that ρ_σ is preserved also by $\rho(t_\beta)$.

Since $g \geq 2$, there exist two distinct vertices $\alpha,\gamma \in \sigma$ such that $i(\alpha,\beta)=1$ and $i(\gamma,\beta)=0$; cf. Figure 7.1. In view of the above remarks, $\rho(\alpha)$ and $\rho(\gamma)$ are invariant under $\rho(t_\alpha)$. Also, $\rho(t_\alpha)$ fixes ρ_σ and ρ_σ is the unique common vertex of edges $\rho(\alpha)$, $\rho(\gamma)$ (by Lemma 12.12). It follows that $\rho(t_\alpha)$ fixes each vertex of $\rho(\alpha)$ and $\rho(\gamma)$. In addition, $\rho(\gamma)$ is invariant under $\rho(t_\beta)$ because $i(\gamma,\beta)=0$. Now, $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$ by Theorem 4.2 and, hence, $\rho(t_\alpha)\rho(t_\beta)\rho(t_\alpha) = \rho(t_\beta)\rho(t_\alpha)\rho(t_\beta)$. Since $\rho(t_\alpha)$ is equal to the identity on $\rho(\gamma)$ and $\rho(\gamma)$ is invariant under $\rho(t_\beta)$, the last equality implies that $\rho(t_\beta)$ also equal to the identity on $\rho(\gamma)$. In particular, $\rho(t_\beta)$ fixes $\rho_\sigma \in \rho(\gamma)$.

Since the configuration C consists of the components of $\mathcal{C}$ and the dual circles (cf. 7.1, we have shown that ρ_σ is fixed by the images under ρ of the Dehn twists along all circles of the configuration $\mathcal{C}$. Hence, the result follows from Theorem 7.3. □

Lemma 13.2. *Suppose that ρ is not almost twist-preserving. Let C be the maximal system of circles introduced in* 7.1, *and let σ be the simplex of $C(S)$ corresponding to C. Then $\rho_\sigma \in \rho(\gamma)$ for all $\gamma \in V_0(S)$.*

Proof. Let $\gamma \in V_0(S)$ and $\alpha \in \sigma$. Since both α and γ are the isotopy classes of nonseparating circles, $f(\alpha)=\gamma$ for some $f \in \mathrm{PMod}_S$. Thus, $f t_\alpha f^{-1} = t_\gamma$ and $\rho(f)(\rho(\alpha)) = \rho(f)(\sigma(\rho(t_\alpha))) = \sigma(\rho(f)\rho(t_\alpha)\rho(f)^{-1}) = \sigma(\rho(f t_\alpha f^{-1})) = \sigma(\rho(t_\gamma)) = \rho(\gamma)$. Because $\rho_\sigma \in \rho(\alpha)$ by the definition of ρ_σ (cf. Lemma 12.12), this implies that $\rho(f)(\rho_\sigma) \in \rho(\gamma)$. On the other hand, $\rho(f)(\rho_\sigma) = \rho_\sigma$ by Lemma 13.1. The lemma follows. □

Lemma 13.3. *Suppose that ρ is not almost twist-preserving. Let C be the maximal system of circles introduced in* 7.1, *and let σ be the simplex of $C(S)$ corresponding to C. Then $\rho(\mathrm{Mod}_S)$ is contained in the stabilizer of ρ_σ in $\mathrm{Mod}_{S'}$.*

Proof. If $\alpha,\beta \in V_0(S)$, $\alpha \neq \beta$ and $i(\alpha,\beta)=0$, then $\rho(\alpha)$ and $\rho(\beta)$ have a unique common vertex by Lemma 12.12 (namely, ρ_τ, where $\tau = \{\alpha,\beta\}$). Lemma 13.2 implies that this common vertex is equal to ρ_σ for all such α,β.

Now, let us choose such a pair α,β. Let $f \in \mathrm{Mod}_S$. Then γ,δ, where $\gamma = f(\alpha), \delta = f(\beta)$, is another such pair. Thus, ρ_σ is the unique common vertex of $\rho(\gamma),\rho(\delta)$. On the other hand, $\rho(f)(\rho(\alpha)) = \rho(f)(\sigma(\rho(t_\alpha))) = \sigma(\rho(f)\rho(t_\alpha)\rho(f)^{-1}) = \sigma(\rho(f t_\alpha f^{-1})) = \sigma(\rho(t_\gamma)) = \rho(\gamma)$ and, similarly, $\rho(f)(\rho(\beta)) = \rho(\delta)$. It follows that $\rho(f)$ maps the unique common vertex of $\rho(\alpha)$ and $\rho(\beta)$ into the unique common vertex of $\rho(\gamma)$ and $\rho(\delta)$. Since both of them are equal to ρ_σ, we have $\rho(f)(\rho_\sigma) = \rho_\sigma$. This completes the proof. □

Lemma 13.4. *Suppose that ρ is not almost twist-preserving. Let σ be the simplex of $C(S)$ considered in Lemmas 13.1—13.3. Let z be some circle on S' in the isotopy class ρ_σ and let $R' = S'_z$ be the result of cutting S' along z.*

(i) *If z is nonseparating, then R' is a connected surface of genus $g' - 1$ with $b' + 2$ boundary components.*

(ii) *If z is nonseparating, then R' is a connected surface of genus $g' - 1$ with $b' + 2$ boundary components. If z is separating, then R' consists of two components. One of them is a disc with two holes, and the other is a connected surface of genus g' with $b' - 1$ boundary components.*

Proof. (i) This part of the lemma is obvious.

(ii) For any $\alpha \in V_0(S)$ we can realize $\rho(\alpha)$ by a system of circles having z as one of the two components, because $\rho_\sigma \in \rho(\alpha)$ by Lemma 13.2 (there are exactly two components by Lemma 12.9). Let us denote by $C(\alpha)$ the other component of this system of circles.

It follows from Lemma 12.8 that $i(\alpha, \beta) = 0$ if and only if $i(C(\alpha), C(\beta)) = 0$. In particular, if $i(\alpha, \beta) \neq 0$, then $i(C(\alpha), C(\beta)) \neq 0$ and, hence, $C(\alpha)$ and $C(\beta)$ are contained in the same component of R'.

Now, note that for any two vertices $\alpha, \beta \in V_0(S)$ there exists a vertex $\gamma \in V_0(S)$ such that $i(\alpha, \gamma) \neq 0$ and $i(\gamma, \beta) \neq 0$. For example, it is sufficient to take $\gamma = f^N(\gamma')$, where f is pseudo-Anosov element, $\gamma' \in V_0(S)$ and N is sufficiently big. It follows that $C(\alpha)$ and $C(\beta)$ are contained in the same component of R' (namely, in the component containing $C(\gamma)$). Thus all circles $C(\alpha), \alpha \in V_0(S)$ are contained in the same component of R'.

Let us consider now some maximal system of nonseparating circles on S and the corresponding simplex τ in $C(S)$. For any two vertices $\alpha, \beta \in \tau$ we have $i(C(\alpha), C(\beta)) = 0$ (because $i(\alpha, \beta) = 0$). Hence, we may assume that circles $C(\alpha), \alpha \in \tau$ are pairwise disjoint. By Lemma 12.12, these circles are pairwise non-isotopic and none of them is isotopic to z. Hence, circles $C(\alpha), \alpha \in \tau$ together with z form a system of circles on S'. It has $3g - 3 + b + 1 = 3g - 2 + b$ components, because τ has $3g - 3 + b$ vertices. On the other hand, $3g - 2 + b = 3g' - 3 + b'$ by Lemma 12.13. Hence, this system of circles is a maximal system of circles on S'. Since all components of this maximal system of circles other than z are contained in the same component of S'_z, the other component of S'_z is a disc with two holes.

So, we proved that one of the components of $R' = S'_z$ is a disc with two holes. Obviously, this implies that the other component has genus g' (the same as S') and $b' - 1$ boundary components. This completes the proof. □

Lemma 13.5. *Let Q be a compact connected orientable surface, c be a nontrivial circle on Q and $R = Q_c$. Let $M(c)$ be the stabilizer in* Mod_Q *of the isotopy class of c.*

(i) *If c is nonseparating, then the kernel of the reduction homorphism $r_c : M(c) \to \mathrm{Mod}_R$ is an infinite cyclic subgroup contained in the center of $M(c)$.*

(ii) *If c divides Q into two parts P and P_0 such that P_0 is a disc with two holes and P is not a disc with two holes, then Mod_R fixes the component P of R and the kernel of the composition $\pi_P \circ r_c : M(c) \to \mathrm{Mod}_P$ is an infinite cyclic subgroup contained in the center of $M(c)$.*

Proof. (i) The kernel of r_c is generated by the Dehn twist t_c about c, which is clearly central in $M(c)$ (cf. 2.3). This proves (i).

(ii) Since P and P_0 are not diffeomorphic, Mod_R fixes both components P and P_0 of R. This implies that $\pi_P \circ r_c$ is well defined (formally, $r_c(M(c)) \subset \mathrm{Mod}_R = \mathrm{Mod}_R(P)$).

Every element of the kernel of $\pi_P \circ r_c$ obviously can be represented by diffeomorphism $F : Q \to Q$ equal to the identity on P. Such a diffeomorphism F is uniquely defined by the induced diffeomorphism $F_0 : P_0 \to P_0$. Clearly, F_0 is equal to the identity on the component c of the boundary ∂P_0, but may interchange two other boundary components. Moreover, any diffeomorphism $P_0 \to P_0$ equal to the identity on c can arise in this way. The group G of such diffeomorphisms $P_0 \to P_0$, considered up to isotopies fixed on c, is known to be infinite cyclic: it is generated by the so-called *half-twist* about c; the square of this generator is a Dehn twist about c. The obvious map from this group G to the kernel of $\pi_P \circ r_c$ is surjective by the above remarks; it is injective because its restriction to the infinite cyclic subgroup of powers of the Dehn twist about c is obviously injective. It follows that the kernel of $\pi_P \circ r_c$ is infinite cyclic.

Finally, any element of $M(c)$ can be represented by a diffeomorphism $F : Q \to Q$ preserving c. Such a diffeomorphism preserves sides of c, because P and P_0 are not diffeomorphic, and, hence, preserves the orientation of c (because F is orientation-preserving). Therefore, replacing F by an isotopic diffeomorphism if necessary, we may assume that F is equal to the identity on c. Now, the description of the kernel of $\pi_P \circ r_c$, given in the previous paragraph (and, in particular, the fact that G is abelian) implies that the isotopy class of F commutes with all elements of this kernel. Hence, the kernel of $\pi_P \circ r_c$ is contained in the center of $M(c)$. □

Lemma 13.6. *Suppose that ρ is not almost twist-preserving. Let σ be the simplex of $C(S)$ considered in Lemmas* 13.1—13.3. *Let z be some circle on S' in the isotopy class ρ_σ and let $R' = S'_z$ be the result of cutting S' along z.*

If z is nonseparating, let $P' = R'$.

If z is separating, then there is a unique component of of R' which is not a disc with two holes. Let us denote by P' this component.

In both cases $\mathrm{Mod}_{R'}$ *fixes the component* P' *of* R' *and thus the induced homomorphism* $\rho' = \pi_{P'} \circ r_z \circ \rho : \mathrm{Mod}_S \to \mathrm{Mod}_{P'}$ *is well defined. Moreover,* ρ' *is an injective almost twist-preserving homomorphism.*

Proof. First, note that $r_z \circ \rho$ is well defined because $\rho(\mathrm{Mod}_S)$ is contained in the stabilizer of ρ_σ by Lemma 13.3.

If z is nonseparating, then $P' = R'$ and $\pi_{P'} = \mathrm{id}$. It follows that $\rho' = r_z \circ \rho$ and so ρ' is well defined.

If z is separating, then one of the components of R' is a disc with two holes by Lemma 13.4. The other component cannot be a disc with two holes, because it has genus g' and $b'-1$ boundary components by Lemma 13.4 and $3g'+b'-1 = 3g+b \geq 6$ by Lemma 12.13 and the assumption $g \geq 2$. So, one component of R' is a disc with two holes and the other is not. It follows that $\mathrm{Mod}_{R'}$ fixes both components of R'. Hence, $\rho' = \pi_{P'} \circ r_z \circ \rho$ is, indeed, well defined.

It remains to prove that ρ' is injective and almost twist-preserving.

The kernel of ρ' is isomorphic via ρ to the intersection of $\rho(\mathrm{Mod}_S)$ with the kernel of $\pi_{P'} \circ r_z$. If z is nonseparating, then $\pi_{P'} \circ r_z = r_z$ and, hence, the kernel of $\pi_{P'} \circ r_z$ is infinite cyclic by Lemma 13.5 (i). If z is separating, then the kernel of $\pi_{P'} \circ r_z$ is infinite cyclic by Lemma 13.5 (ii). It follows that the kernel of ρ' is a subgroup of an infinite cyclic group and thus is either trivial or infinite cyclic group. Since no infinite cyclic subgroup of Mod_S can be normal (this follows easily from the Thurston's classification of elements of Mod_S; alternatively, one may use [I3], Exersices 5.a, 5.b and Lemma 9.12), the kernel of ρ' is trivial.

So, ρ' is injective. If $\alpha \in V_0(S)$, then $\rho(t_\alpha^n)$ is a multitwist about $\rho(\alpha)$ for some $n \neq 0$; cf. 12.7. Since $\rho(\alpha)$ consists of two vertices and one of them is the isotopy class ρ_σ of z (by Lemma 13.2), it follows that $r_z(\rho(t_\alpha^n))$ is a power of a Dehn twist (about a circle representing the other vertex). Hence, $\pi_{P'}(r_z(\rho(t_\alpha^n)))$ is a power of a Dehn twist. This proves that $\rho' = \pi_{P'} \circ r_z \circ \rho$ is almost twist-preserving and, hence, completes the proof. □

Theorem 13.7. *Let* S *and* S' *be compact connected orientable surfaces. Suppose that the genus of* S *is at least* 2 *and* S' *is not a closed surface of genus* 2*. Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. If* $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ *is an injective homorphism, then* ρ *is induced by a diffeomorphism* $S \to S'$.

Proof. In view of Theorem 10.9, it is sufficient to consider the case when ρ is not almost twist-preserving.

Let σ be the simplex of $C(S)$ considered in Lemmas 13.1—13.4 and 13.6. As in Lemma 13.6, let z be some circle on S' in the isotopy class ρ_σ and let $R' = S'_z$. Let P' be the component of R' introduced in Lemma 13.6. By Lemma 13.6, the homomorphism $\rho' = \pi_{P'} \circ r_z \circ \rho : \mathrm{Mod}_S \to \mathrm{Mod}_{P'}$ is well defined and is an injective almost twist-preserving homomorphism. In addition, Lemma 13.4 implies that the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{P'}$ is one less than the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ and, hence, is equal to the maxima of ranks of abelian subgroups of Mod_S (it cannot be less than this maxima for Mod_S, because ρ' is injective). This means, in particular, that Theorem 10.9 applies to ρ' and implies that ρ' is induced by some diffeomorphism $H : S \to P'$.

In the remaining part of the proof we will use the notations (and the results) of Section 5.

If $F : S \to S'$ is a diffeomorphism fixed on ∂S, then the diffeomorphism $H \circ F \circ H^{-1} : P' \to P'$ gives rise to a diffeomorphism $S' \to S'$ by the glueing in the case when z is nonseparating (note the $H \circ F \circ H^{-1}$ is fixed on $\partial P'$) and by the extending by the identity in the case when z is separating. By passing to the isotopy classes, we get a homomorphism $\mathcal{M}_S \to \mathcal{M}_S$ induced by H. We will denote it by H_*. (Compare the proof of Theorem 8.9.) By choosing the orientations of S and S' appropriately, we may assume that H is orientation-preserving. Then $H_*(\tilde{t}_c) = \tilde{t}_{H(c)}$ for any circle c on S. As in the proof of Theorem 8.9, let us consider the following diagram.

$$\begin{array}{ccc} \mathcal{M}_S & \xrightarrow{H_*} & \mathcal{M}_{S'} \\ \Big\downarrow p & & \Big\downarrow p' \\ \mathrm{PMod}_S & \xrightarrow{\rho} & \mathrm{Mod}_{S'} \end{array}$$

The vertical maps are the canonical homomorphisms $p : \mathcal{M}_S \to \mathrm{PMod}_S$, $p' : \mathcal{M}_{S'} \to \mathrm{Mod}_{S'}$. Note that we cannot claim that this diagram is commutative. But in fact it is quite close to being commutative. Namely, the compositions $\pi_{P'} \circ r_z \circ p' \circ H_*$ and $\pi_{P'} \circ r_z \circ \rho \circ p : \mathcal{M}_S \to \mathrm{Mod}_{P'}$ are equal, because $\rho' = \pi_{P'} \circ r_z \circ \rho$ is induced by H. Therefore, $p' \circ H_*(f)$ and $\rho \circ p(f)$ differ by an element of the kernel of $\pi_{P'} \circ r_z$ for any $f \in \mathcal{M}_S$. By Lemma 13.5, this kernel is an infinite cyclic group, contained in the center of $M(z)$, where $M(z)$ is the stabilizer in $\mathrm{Mod}_{S'}$ of the isotopy class of z (i.e., of ρ_σ). We will denote this kernel by K.

Let us consider the Dehn twist $\tilde{t}_c \in \mathcal{M}_S$, where c is a boundary component of S corresponding to z under H. By Theorem 5.3 the element $\tilde{t}_c \in \mathcal{M}_S$ belongs to the commutator subgroup of $\mathcal{M}_S$. (Here we use the assumption that the genus of S is at least 2 in a crucial way.) In other words,

$$\tilde{t}_c = \prod_{i=1}^{n} [f_i, g_i],$$

for some $f_i, g_i \in \mathcal{M}_S$, $1 \le i \le n$, where $[f, g]$ denotes the commutator $fgf^{-1}g^{-1}$. Now,

$$t_z = p'(\tilde{t}_z) = p'(\tilde{t}_{H(c)}) = p'(H_*(\tilde{t}_c)) =$$

$$= p' \circ H_*(\tilde{t}_c) = p' \circ H_*(\prod_{i=1}^{n} [f_i, g_i]) =$$

$$= \prod_{i=1}^{n} [p' \circ H_*(f_i), p' \circ H_*(g_i)] =$$

$$= \prod_{i=1}^{n} [\rho \circ p(f_i) k_i, \rho \circ p(g_i) l_i]$$

for some $k_i, l_i \in K$, $1 \le i \le n$ by the discussion in the previous paragraph. Since $\rho(\mathrm{PMod}_S)$ is contained in $M(z)$ by Lemma 13.1 and K is contained in the center of $M(z)$, as we noticed above, the last expression is equal to

$$\prod_{i=1}^{n} [\rho \circ p(f_i), \rho \circ p(g_i)] = \rho \circ p(\prod_{i=1}^{n} [f_i, g_i]) =$$

$$= \rho \circ p(\tilde{t}_c) = \rho(1) = 1$$

(since c is a boundary component, $p(\tilde{t}_c) = 1$). We conclude that $t_z = 1$, contradicting to the fact that z is a nontrivial circle.

Thus the assumption that ρ is not almost twist-preserving leads to a contradiction. In view of the above, this completes the proof. $\square$

Theorem 13.8. *Let S be a compact connected orientable surface of genus at least 2 with one boundary component. Then the natural surjective homomorphism $\mathcal{M}_S \to \mathrm{PMod}_S$ is nonsplit, where $\mathcal{M}_S$ is the group of isotopy classes of diffeomorphisms $S \to S$ fixed on ∂S, with respect to isotopies fixed on ∂S (cf. Section 5).*

Proof. Note that $\mathrm{PMod}_S = \mathrm{Mod}_S$, because S has one boundary component. If $\mathcal{M}_S \to \mathrm{PMod}_S$ splits, then there is an injective homomorphism $\theta : \mathrm{Mod}_S \to \mathcal{M}_S$.

Let S' be the surface obtained from S by attaching a disc Q with two holes to ∂S along a component C of ∂Q. Then S' is not a closed surface of genus 2 and the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by one.

Each diffeomorphism $F : S \to S$ which is fixed on ∂S extends by the identity to a diffeomorphism $F' : S' \to S'$. This extension also applies to isotopies of S which are fixed on ∂S. Hence, it induces a natural homomorphism $\eta : \mathcal{M}_S \to \mathrm{Mod}_{S'}$. It is well known that η is injective. (This can be easily established by reducing along C.) Now, the composition $\eta\theta$ is an injective homomorphism $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$. But, by Theorem 13.7, this is impossible. Hence, the homomorphism $\mathcal{M}_S \to \mathrm{PMod}_S$ is nonsplit. $\square$

14. Proof of Theorem 2

In this section we deduce Theorem 2 of the Introduction from Theorem 13.7 (i.e., from Theorem 3 of the Introduction). Cf. Theorem 14.2.

Lemma 14.1. *Let S, S' be a pair of surfaces satisfying the assumptions of the first paragraph of Section 12. If $\rho : \mathrm{Mod}^*_S \to \mathrm{Mod}^*_{S'}$ is an injective homomorphism, then $\rho(\mathrm{Mod}_S) \subset \mathrm{Mod}_{S'}$.*

Proof. Suppose that, to the contrary, $\rho(\mathrm{Mod}_S)$ is not contained in $\mathrm{Mod}_{S'}$. Since Mod_S is generated by the Dehn twists about nonseparating circles, in this case $\rho(t_\alpha) \in \mathrm{Mod}^*_{S'} \setminus \mathrm{Mod}_{S'}$ for some $\alpha \in V_0(S)$. In other words, $\rho(t_\alpha)$ is an orientation-reversing element for some $\alpha \in V_0(S)$. Let us fix such an α.

Suppose first that ρ is almost twist-preserving. Then $\rho(t_\alpha^N) = t^M_{\rho(\alpha)}$ for some $M, N \neq 0$. Clearly, $\rho(\alpha) = \sigma(\rho(t_\alpha^N)) = \sigma(\rho(t_\alpha t_\alpha^N t_\alpha^{-1})) = \sigma(\rho(t_\alpha)\rho(t_\alpha^N)\rho(t_\alpha)^{-1}) = \rho(t_\alpha)(\sigma(\rho(t_\alpha^N))) = \rho(t_\alpha)(\rho(\alpha))$. Thus $\rho(t_\alpha)$ preserves the isotopy class $\rho(\alpha)$ and, since $\rho(t_\alpha)$ is orientation-reversing, we have $\rho(t_\alpha)t^M_{\rho(\alpha)}\rho(t_\alpha)^{-1} = t^{-M}_{\rho(\alpha)}$. On the other hand, $\rho(t_\alpha)$ obviously commutes with $\rho(t_\alpha^N) = t^M_{\rho(\alpha)}$. Hence, we have a contradiction in this case and, so, ρ cannot be almost twist-preserving.

If ρ is not almost twist-preserving, then $\rho(\gamma)$ is an edge of $C(S')$ for any $\gamma \in V_0(S)$ in view of Lemma 12.9. Let us choose $\beta \in V_0(S)$ such that $i(\beta, \alpha) = 0$ and $\beta \neq \alpha$. Then $\rho(\alpha)$ and $\rho(\beta)$ have a unique common vertex by Lemma 12.10. Since $\rho(t_\alpha)$ commutes with both $\rho(t_\alpha^N)$ and $\rho(t_\beta^N)$ for any N, the image $\rho(t_\alpha)$ preserves both $\rho(\alpha)$ and $\rho(\beta)$ and, hence, fixes the unique common vertex of $\rho(\alpha)$ and $\rho(\beta)$. It follows that $\rho(t_\alpha)$ fixes all the vertices of $\rho(\alpha)$ and $\rho(\beta)$ (because $\rho(\alpha)$ and $\rho(\beta)$ are edges). Now, $\rho(t_\alpha^n)$ is a multitwist about $\rho(\alpha)$ for some $n \neq 0$ (cf. 12.7). Thus $\rho(t_\alpha^n) = t_\gamma^l t_\delta^m$ for some $l, m \neq 0$, where γ, δ are the vertices of the edge $\rho(\alpha)$. Because $\rho(t_\alpha)$ fixes both γ and δ and $\rho(t_\alpha)$ is orientation-reversing, we have $\rho(t_\alpha)t_\gamma^l t_\delta^m \rho(t_\alpha)^{-1} = t_\gamma^{-l} t_\delta^{-m}$. On the other hand, $\rho(t_\alpha)$ obviously commutes with $\rho(t_\alpha^n) = t_\gamma^l t_\delta^m$. We again reached a contradiction. This completes the proof. □

Theorem 14.2. *Let S be a closed orientable surface of genus at least 2. Then there is no injective homomorphisms $\mathrm{Out}(\pi_1(S)) \to \mathrm{Aut}(\pi_1(S))$. In particular, the natural epimorphism $\mathrm{Aut}(\pi_1(S)) \to \mathrm{Out}(\pi_1(S))$ is nonsplit.*

Proof. Let x be a basepoint on S. Since the genus of S is at least two, the center of $\pi_1(S, x)$ is trivial. Hence, we have a short exact sequence:

$$1 \to \pi_1(S, x) \xrightarrow{\partial} \mathrm{Aut}(\pi_1(S, x)) \to \mathrm{Out}(\pi_1(S, x)) \to 1,$$

in which ∂ maps an element $a \in \pi_1(S, x)$ into the inner automorphism $g \mapsto aga^{-1}$.

Let $\mathrm{Diff}(S)$ denote the group of diffeomorphisms $S \to S$ and let $\mathrm{Diff}(S, x)$ denote the group of diffeomorphisms $S \to S$ fixing the basepoint x. Let S' be a compact connected orientable surface of the same genus as S and with one boundary component. Clearly, there exists a map $(S', \partial S') \to (S, x)$ inducing a diffeomorphism $S' \setminus \partial S' \to S \setminus \{x\}$ and collapsing the boundary $\partial S'$ to the point x. By using such a map we may identify S with the surface obtained from S' by blowing down the boundary $\partial S'$ to a point. This identifucation induces a natural isomorphism $\mathrm{Mod}^*_{S'} \to \pi_0(\mathrm{Diff}(S, x))$. In addition, $\mathrm{Mod}^*_S = \pi_0(\mathrm{Diff}(S))$, simply by the definition.

According to Theorem 4.3 of [B], we have a short exact sequence:

$$1 \to \pi_1(S, x) \xrightarrow{\partial} \pi_0(\mathrm{Diff}(S, x)) \to \pi_0(\mathrm{Diff}(S)) \to 1.$$

The kernel of the homomorphism $\pi_0(\mathrm{Diff}(S,x)) \to \pi_0(\mathrm{Diff}(S))$ corresponds to diffeomorphisms $F : (S,x) \to (S,x)$ which are isotopic to the identity, but not necessarily by an isotopy fixing x. Under such an isotopy, x travels around a loop γ in S. The inclusion $\pi_1(S,x) \to \pi_0(\mathrm{Diff}(S,x))$ maps the homotopy class of γ to the isotopy class of F (with the basepoint x assumed to be fixed during the corresponding isotopies) and the action of F on $\pi_1(S,x)$ is given by the conjugation by the homotopy class of γ. The composition of the natural homomorphism $\pi_0(\mathrm{Diff}(S,x)) \to \mathrm{Aut}(\pi_1(S))$ with the natural homomorphism $\mathrm{Aut}(\pi_1(S)) \to \mathrm{Out}(\pi_1(S))$ factors through $\pi_0(\mathrm{Diff}(S,x)) \to \pi_0(\mathrm{Diff}(S))$ to provide a natural homomorphism $\pi_0(\mathrm{Diff}(S)) \to \mathrm{Out}(\pi_1(S))$. According to the Baer-Nielsen Theorem (cf., for example, [Z] Corollary 11.7), $\pi_0(\mathrm{Diff}(S)) \to \mathrm{Out}(\pi_1(S))$ is an isomorphism. It follows that, as is well known, $\mathrm{Mod}^*_S = \pi_0(\mathrm{Diff}(S))$ is naturally isomorphic to $\mathrm{Out}(\pi_1(S))$.

The above discussion is summarized by the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S,x) & \xrightarrow{\partial} & \pi_0(\mathrm{Diff}(S,x)) & \longrightarrow & \pi_0(\mathrm{Diff}(S)) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \pi_1(S,x) & \xrightarrow{\partial} & \mathrm{Aut}(\pi_1(S,x)) & \longrightarrow & \mathrm{Out}(\pi_1(S,x)) & \longrightarrow & 1
\end{array}$$

(the commutativity of the square involving the maps ∂ follows from the description of these maps given above). By the preceding discussion, the horizontal rows of this diagram are exact and the two left-hand and the two right hand vertical arrows are isomorphisms. Hence, the vertical arrow in the middle of the diagram is also an isomorphism. Thus, we have a natural identification of the homomorphism $\mathrm{Aut}(\pi_1(S)) \to \mathrm{Out}(\pi_1(S))$ with the homomorphism $\pi_0(\mathrm{Diff}(S,x)) \to \pi_0(\mathrm{Diff}(S))$. In view of the above, the last homomorphism can be identified with a homomorphism $\mathrm{Mod}^*_{S'} \to \mathrm{Mod}^*_S$.

Suppose that there exist an injective homomorphism $\mathrm{Out}(\pi_1(S)) \to \mathrm{Aut}(\pi_1(S))$. By the previous discussion, this means that there is an injective homomorphism $\mathrm{Mod}^*_S \to \mathrm{Mod}^*_{S'}$. By Lemma 14.1, it maps Mod_S to $\mathrm{Mod}_{S'}$. Hence, there exist an injective homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$.

Now, S is a closed surface of genus at least two and S' is a surface of the same genus with one boundary component. In particular, S' is not a closed surface of genus two and the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by one. Hence, Theorem 13.7 implies that there exists a diffeomorphism $S \to S'$. The obvious contradiction proves that there are no injective homomorphisms $\mathrm{Out}(\pi_1(S)) \to \mathrm{Aut}(\pi_1(S))$. $\square$

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