The Lax Proof of the Change of Variables Formula, Differential Forms, a Determinantal Identity, and Jacobi Multipliers

Nikolai V. Ivanov

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Introduction. In a beautiful paper [Lax1], P. Lax presented an elementary proof of a special case of the change of variables theorem. As explained in [Lax1], this special case is sufficient to give a very simple proof of the Brouwer fixed point theorem. In [Lax2], Lax explained how one can deduce the general case of the change of variables theorem from this special case using some standard tools, like the partition of unity, from the no-man's-land between advanced calculus and the three great differential theories (differential topology, differential geometry, ordinary differential equations), to paraphrase S. Lang a little. (See [Lang1], Foreword.)

The first goal of this note is to present a differential forms version of the Lax proof of (a special case of) the change of variables formula. We attempted to follow the Lax arguments as closely as possible. A special care was taken to be completely explicit about all results concerned with the integration of differential forms we use, since the usual expositions assume (or prove in a classical way) the change of variables formula at the very beginning of theory of integration of differential forms. One of the exceptions is Lang's book [Lang2], where the differential part of the theory of differential forms is clearly separated from the integration (because the first one makes sense in infinite dimensions, and one of the goals of Lang is to work in infinite dimensions whenever possible). See [Lang2], Chapter V, especially §3.

Our second goal is to present a fairly detailed comparison of our proof with the Lax one. Such a comparison is very instructive; it sheds a light

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on both the efficiency of the differential form theory and the brilliance with which Lax uses the classical analysis. When possible, we will keep Lax notations and terminology.

A key role in the Lax proof is played by a fairly mysterious determinantal identity; see the identity (3) below. It is almost invisible in the differential forms version of the proof. In fact, the identity (3) plays a similar role in at least one other proof of the change of variables formula, and it is ubiquitous in analytical approaches to the Brouwer fixed point theorem and related topics. We will quickly review the relevant papers. With the exception of 1910 paper of J. Hadamard [Had], none of them mentions the original context in which this identity appeared, namely, the Jacobi theory of the multipliers for systems of ordinary differential equations. This theory generalizes the well known theory of the integrating factors (due to Euler). It turns out that differential forms provide a very good framework for this theory, and in the last part of the paper we will present, using the language of differential forms, the part of this theory directly related to the identity (3).

In this note we need only a small fragment of the theory of differential forms; all what is needed can be found, for example, in Chapter 2 of F. Warner's textbook [W]. One may also suggest a much more elementary textbook by H. Edwards [Ed]. Other textbooks include V. Guillemin and A. Pollack [G-P] and S. Lang [Lang2] (the last one is more abstract and advanced than others, partially because it deals with not necessarily finitely dimensional situation from the very beginning). I would like also to recommend to the reader the nice article by H. Samelson [S2], summarizing both the theory of differential forms and its history (which contains some surprises).

A version of the Lax proof similar to the ours one was suggested by M. Taylor [T2]. He also uses differential forms, but his proof differs from the Lax one in some other respects also. In contrast with this note, he aims at some more general versions of the change of variables theorem (assuming less regularity from the change of variables mapping). Apparently, our version is equally suitable for generalizations, but the goals of this note are purely expository.

The special case of the change of variables formula considered by [Lax1] deals with the following situation. Let $\varphi(x) = y$ be a mapping of the n-dimensional x space into n-dimensional y space, i.e, let φ be a mapping $\mathbf{R}^n \to \mathbf{R}^n$. We assume that:

(i) φ is once differentiable;

(ii) φ is the identity outside some sphere, say the unit sphere:

$$\varphi(x) = x \text{ for } |\,x\,| \geq 1.$$

Change of variable theorem. *Let f be a continuous function of compact support. Then*

$$\int f(\varphi(x))J(x)\,dx = \int f(y)\,dy,$$

where J is the Jacobian determinant of the mapping φ:

$$J(x) = \det \frac{\partial \varphi_j}{\partial x_i};$$

here φ_j is the j^{th} component of φ.

We will start our proof with a simple lemma.

Lemma. *Let g be a once differentiable function on the n-dimensional y space; $g : \mathbf{R}^n \to \mathbf{R}$. Let $\psi = (\psi_1, \ldots, \psi_n)$ be a once differentiable map $\mathbf{R}^n \to \mathbf{R}^n$. Let $1 \leq i \leq n$. Then*

$$d(g \circ \psi) \wedge d\psi_1 \wedge \ldots \widehat{d\psi_i} \wedge \ldots \wedge d\psi_n = (-1)^{i-1} \frac{\partial g}{\partial y_i} \circ \psi \, d\psi_1 \wedge \ldots \wedge d\psi_n,$$

where the roof $\widehat{\ }$ indicates the omitted term.

Proof. Note that

$$d(g \circ \psi) = d(\psi^*(g)) = \psi^*(dg) =$$

$$= \psi^*(\sum_{j=1}^{n} \frac{\partial g}{\partial y_j}\, dy_j) = \sum_{j=1}^{n} (\frac{\partial g}{\partial y_j}) \circ \psi \, d\psi_j.$$

Hence

$$d(g \circ \psi) \wedge d\psi_1 \wedge \ldots \widehat{d\psi_i} \wedge \ldots \wedge d\psi_n =$$

$$= (\sum_{j=1}^{n} (\frac{\partial g}{\partial y_j}) \circ \psi \, d\psi_j) \wedge d\psi_1 \wedge \ldots \widehat{d\psi_i} \wedge \ldots \wedge d\psi_n =$$

$$= (-1)^{i-1} (\frac{\partial g}{\partial y_i}) \circ \psi \, d\psi_1 \wedge \ldots \wedge d\psi_n,$$

because $d\psi_j \wedge d\psi_i = 0$ for $j \neq i$. The lemma follows. (Note that the sign $(-1)^{i-1}$ comes from the need to move $d\psi_i$ from first place to the i^{th} place.)

Corollary. *Let h be a once differentiable function on the n-dimensional x space; $h : \mathbf{R}^n \to \mathbf{R}$. Let $1 \leq i \leq n$. Then*

$$d(h\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \ldots \wedge dx_n) =$$
$$= dh \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \ldots \wedge dx_n =$$
$$(-1)^{i-1} \frac{\partial h}{\partial x_i}\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

Proof. The first equality is simply the definition of the exterior derivative. In order to prove the second one, just replace x by y and apply the lemma to h equal to g and ψ equal to the identity map. The Corollary follows.

We will need the following special case of the Stokes' theorem. Let $c > 0$, and let I be the c-cube in the n-dimensional x space $\mathbf{R}^n$, i.e. I is given by the inequalities

$$|\, x_i| \leq c, \quad i = 1, 2, \ldots, n.$$

Stokes' theorem. *Let ω be an $(n-1)$-form on $\mathbf{R}^n$,*

$$\omega = \sum_{i=1}^{n} h_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n.$$

Then

$$\int_I d\omega = \int_{\partial I} \omega,$$

where ∂I is the boundary of I.

The integrals here can be understood in a naive sense, without any recourse to the general theory of integration of differential forms. First of all, $d\omega$ is an n-form on $\mathbf{R}^n$, and the integral $\int_I \psi$ of an n-form $\psi = h\, dx_1 \wedge \ldots \wedge dx_n$ can be defined simply as the usual integral $\int_I h$. In order to avoid any discussion of the orientation of I and of the induced orientation of ∂I, we will understand the integral $\int_{\partial I} \omega$ simply as a shorthand for

$$\sum_{i=1}^{n} (-1)^{i-1} \Big(\int_{I_i^+} h_i - \int_{I_i^-} h_i\Big),$$

where I_i^+ and I_i^- are the faces of c-cube I given by the equations $x_i = c$ and $x_i = -c$ respectively. Note that the integral defined in such a way is clearly linear with the respect of the addition of the differential forms.

The possibility to restrict ourselves by such a naive and limited version of the Stokes' theorem is a great advantage of the Lax approach. After this limited version is used to prove the above change of variables formula and the formula is extended to the general case following [Lax2], one can return to the theory of integration of differential forms in full generality (where it requires the general version of the change of variables formula) and easily prove the Stokes' theorem in any desired generality (say, for manifolds with boundary instead of I, if one already has the notion of a manifold at hand).

Proof of the Stokes' theorem. It is sufficient to deal with the different summands of ω separately. Let $\omega_i = h_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$. By Corollary,

$$d\omega_i = (-1)^{i-1}\frac{\partial h_i}{\partial x_i}\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

Hence,

$$\int_I d\omega = \int_I (-1)^{i-1}\frac{\partial h_i}{\partial x_i} = (-1)^{i-1}\int\Big(\int \frac{\partial h_i}{\partial x_i}\, dx_i\Big)\, dx_1 \ldots \widehat{dx_i} \ldots dx_n =$$

$$= (-1)^{i-1}\Big(\int_{I_i^+} h_i\, dx_1 \ldots \widehat{dx_i} \ldots dx_n - \int_{I_i^-} h_i\, dx_1 \ldots \widehat{dx_i} \ldots dx_n\Big) =$$

$$= (-1)^{i-1}\Big(\int_{I_i^+} h_i - \int_{I_i^-} h_i\Big) = \int_{\partial I} \omega_i.$$

The last step uses our definition of the integral $\int_{\partial I}$. Before we used the Fubini theorem (twice) and the fundamental theorem of calculus. (Of course, the Stokes' theorem is simply the multidimensional form of the fundamental theorem of calculus.) This proves the theorem.

The above proof is very close to the proof in [Lang2], Chapter XVII, §1.

Now, we are ready for the proof of the change of variables formula.

Proof of the Change of variable theorem. It is sufficient to prove the theorem for functions f that are once differentiable and for mappings φ that are twice differentiable, since functions and mapping can be approximated by differentiable ones; see [Lax1].

Following Lax, define

$$g(y_1, y_2, \ldots, y_n) = \int_{-\infty}^{y_1} f(z, y_2, \ldots, y_n)\, dz.$$

The integral is well defined because f has compact support. Clearly, $\partial g/\partial y_1 = f$. The function g is once differentiable together with f. Let $c > 0$ and let I be the c-cube as above. We can choose c to be so large that the support of f is contained in I and the unit ball $\{y : |y| \leq 1\}$ is contained in I. Then $g(y_1, \dots, y_n) = 0$ when $|y_j| \geq c$ for any $j \neq 1$ and when $y_1 \leq -c$. In addition, since φ is equal to the identity outside of the unit ball, $f(\varphi(x))$ is equal to 0 outside I. It follows that we can restrict integration in the theorem to the c-cube I.

Now, let $\varphi = (\varphi_1, \dots \varphi_n)$. Consider the integrand of the left hand side of the change of variables formula:

$$f(\varphi(x))J(x)\, dx_1 \wedge \dots \wedge dx_n = f(\varphi(x))\, d\varphi_1 \wedge \dots \wedge d\varphi_n =$$
$$= \frac{\partial g}{\partial y_1} \circ \varphi\, d\varphi_1 \wedge \dots \wedge d\varphi_n.$$

By Lemma (applied to the case $i = 1$), the last expression is equal to

$$d(g \circ \varphi) \wedge d\varphi_2 \wedge \dots \wedge d\varphi_n.$$

Now, obviously,

$$d(g \circ \varphi) \wedge d\varphi_2 \wedge \dots \wedge d\varphi_n = d(g \circ \varphi\, d\varphi_2 \wedge \dots \wedge d\varphi_n). \tag{1}$$

(Here we implicitly used our assumption that φ is twice differentiable.) Next, we get by the integrating

$$\int_I f(\varphi(x))J(x)\, dx_1 \wedge \dots \wedge dx_n = \int_I d(g \circ \varphi\, d\varphi_2 \wedge \dots \wedge d\varphi_n).$$

By the Stokes' theorem the last expression is equal to

$$\int_{\partial I} g \circ \varphi\, d\varphi_2 \wedge \dots \wedge d\varphi_n.$$

Notice that the boundary ∂I is entirely contained in the domain where φ is equal to the identity mapping. Hence the last integral is equal to

$$\int_{\partial I} g\, dy_2 \wedge \dots \wedge dy_n = \int_{I_1^+} g\, dy_2 \dots dy_n =$$
$$= \int_{I_1^+} \int_{-\infty}^{c} f\, dy_1\, dy_2 \dots dy_n = \int_{I_1^+} \int_{-c}^{c} f\, dy_1\, dy_2 \dots dy_n =$$
$$= \int_I f\, dy_1 dy_2 \dots dy_n = \int f(y)\, dy.$$

In the first equality we used the fact that g is equal to 0 on all faces of I with the exception of I_1^+. This completes the proof of the Change of variable theorem.

Comparison with the Lax proof. Now we are going to compare our proof with the Lax one. We hope that the reader is familiar with the Lax paper. Most of the following discussion can be understood, but cannot be fully appreciated without this. First of all, our proof is based on the same key idea of introducing the function g as the Lax one. We also using the same tool of restricting the integration to a large c-cube I. Our Lemma in the special case $i = 1$ is the Observation of Lax (see [Lax1], p.498) expressed in the language of differential forms, with his φ corresponding to our ψ. Note that the standing assumption on φ in [Lax1], the same as ours, play no role in this Observation (except the differentiability). We proved this Lemma for all i, because this is no more difficult than the case $i = 1$ and this provided us with our Corollary, which in turn was a key ingredient in our proof of the Stokes' theorem.

One may observe that Lax does not uses the Stokes' theorem, at least explicitly. Instead he uses the integration by parts, which is not used in our proof, again at least explicitly, and a determinantal identity (the formula (2.10) in [Lax1]).

Recall that in the calculus of one variable the integration by parts is nothing else as a combination of the Leibniz rule with the fundamental theorem of calculus. The Leibniz rule naturally generalizes to the calculus of differential forms; it is the following formula:

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^a \omega_1 \wedge d\omega_2,$$

where a is the degree of the form ω_1. The several variables version of the fundamental theorem of calculus is the Stokes' theorem, as we already noticed. So, the differential form version of the integration by parts in the case of integration over I (the only case we need) is the following computation:

$$\int_I d\omega_1 \wedge \omega_2 + (-1)^a \omega_1 \wedge d\omega_2 = \int_I d(\omega_1 \wedge \omega_2) = \int_{\partial I} \omega_1 \wedge \omega_2.$$

Now, it seems that we did not used the Leibniz formula either. In fact, its use is hidden in the formula (1). In more details, we can deduce (1) as follows. Notice that

$$d(g \circ \varphi \, d\varphi_2 \wedge \ldots \wedge d\varphi_n) = d(g \circ \varphi) \wedge d\varphi_2 \wedge \ldots \wedge d\varphi_n + g \circ \varphi \, d(d\varphi_2 \wedge \ldots \wedge d\varphi_n)$$

by the Leibniz formula with ω_1 equal to the 0-form $g \circ \varphi$ and ω_2 equal to the form $d\varphi_2 \wedge \ldots \wedge d\varphi_n$. The second summand is zero because

$$d(d\varphi_2 \wedge \ldots \wedge d\varphi_n) = 0, \qquad (2)$$

and this implies (1). From the point of view of differential forms (2) is obvious; a formal proof follows from the Leibniz rule for an $(n-1)$-fold product of forms and the fact that $d(d\varphi_i) = 0$ (here we need φ_i to be twice differentiable). A more geometric proof is presented in [I] (see proof of Lemma in [I]). Expressed in the classical language, the last formula turns into a not quite trivial and fairly mysterious determinantal identity. Let

$$d\varphi_2 \wedge \ldots \wedge d\varphi_n = \sum_{i=1}^{n} A_i \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$

for some A_i. (This formula corresponds to expanding the determinant (2.5) according to the first column in the Lax proof.) Then

$$d(d\varphi_2 \wedge \ldots \wedge d\varphi_n) = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial A_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n$$

(by our Corollary applied to functions A_i in the role of h). As we saw, the left hand side of this equation is equal to 0, and this implies the identity

$$\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial A_i}{\partial x_i} = 0. \tag{3}$$

A_i's are, in fact, determinants of some matrices of partial derivatives of φ. More precisely, A_i is i^{th} minor (of size $(n-1) \times (n-1)$) of the $(n-1) \times n$ matrix $(\partial \varphi_i / \partial x_j)$, where $2 \leq i \leq n$, $1 \leq j \leq n$, as the reader may easily check. This means that $(-1)^{i-1} A_i$ is the exactly the cofactor M_i from the Lax's paper, and our identity is equivalent to his determinantal identity (2.10)

$$\partial_{x_1} M_1 + \cdots + \partial_{x_n} M_n \equiv 0.$$

To sum up, formula (1) hides in it a special case of the Leibniz rule and the above determinantal identity (3). Integrating of this formula and applying the Stokes' theorem to the result amounts to combining the integration by parts with the Lax determinantal identity (2.10).

The remaining parts of both proofs, namely the computation of

$$\int_{\partial I} g \, dy_2 \wedge \ldots \wedge dy_n$$

in our proof and the computation of the boundary term in the integration by parts formula in the Lax proof, are exactly the same.

The identity (3) in the proofs of the Change of Variables formula and the Brouwer Fixed-Point theorem. It is clear that the identity (3) plays an essential role in the Lax proof [Lax1] of the change of variables formula (although, of course, the use of (3) is not the main idea of the proof). The same identity plays a similar role in the proof of the change of variables formula given by H. Leinfelder and Ch. G. Simader [L-S]. We would like to note that the paper of H. Leinfelder and Ch. G. Simader [L-S] also admits a fairly close translation into the language of differential forms, and such translation clarifies their arguments a great deal. The reader may attempt such a translation as an exercise (the note [I] may be of some help).

It turns out that one can use the change of variables formula or some arguments used in its proof to give a short proof of the Brouwer fixed-point theorem. Apparently, the first such proof is due to H. Leinfelder and Ch. G. Simader [L-S], who used their main Lemma to prove both the change of variables formula and the Brouwer fixed-point theorem. L. Báez-Duarte [B] deduced the Brouwer fixed-point theorem directly from the change of variables formula, and his deduction was used by P. Lax [Lax1]. For a topologist, the connection between the change of variables formula and the Brouwer theorem is not very surprising: the change of variables formula is the basis of the functoriality of the de Rham cohomology (cf. [I]). Still, this deduction provides a remarkable shortcut through quite a lot of the machinery.

There are several analytic proofs of the Brouwer theorem. With essentially only one exception, they are similar in spirit, and, what is really surprising, all use the identity (3) in an essential manner. The oldest analytic proof is due to J. Hadamard [Had]. He uses a more general and complicated version of the identity (3). The most well-known proof is, probably, the proof by Dunford and Schwartz; see [D-Sch], p. 467. The identity (3) is the starting point of their proof (see [I] for a discussion). Y. Kannai [K] gives a simpler proof of the Brouwer theorem in the same spirit as the Dunford-Schwartz one, and the identity (3) plays an essential role in it. It plays an essential role in the E. Heinz [Heinz] approach to the theory of the Brouwer degree (which includes the Brouwer fixed-point theorem). Very recently, L. Evans [Ev] presented a version of the Dunford-Schwartz proof; see [Ev], Section 8.1.4. The identity (3) shows up as the Lemma in the Section 8.1.4.b. The Dunford-Schwartz proof inspired a differential forms proof by H. Samelson [S1]; in his exposition the identity (3) is hidden, as one may expect. At the end of his paper [K] Y. Kannai presents a differential forms version of his proof, which turns out to be exactly Samelson's [S1] proof, and the identity (3) is also hidden. M. Taylor [T1] (see [T1], Chapter 1, Section 19) presents a more general version of the Samelson–Kannai proof; the identity (3) appears in the form of the computation $d\varphi^*\omega = \varphi^* d\omega = 0$, where ω is a volume form on an $(n-1)$ manifold and φ is a map with values in this manifold. If $\omega = dx_2 \wedge \ldots \wedge dx_n$ in some local coordinates $(x_2, \ldots, x_n)$, then $d\varphi^*\omega = 0$ is exactly the formula (2). The identity (3) is also hidden in the Taylor's differential forms version [T2] of the Lax proof of the change of variables; and it is hidden in our version of the Lax proof.

The only exceptions are the J. Milnor's proof [M] of the Brouwer theorem and the C. Rogers' [R] version of it. Milnor's proof is strange and completely different from any other proof (the key step is based on the fact that $(\sqrt{(1+t^2)})^n$ is not a polynomial for the odd n). In addition to the Brouwer theorem, Milnor proof gives also the "Hairy Ball Theorem", which is more difficult. Rogers' version [R] keeps the main ideas of Milnor but is "less strange". In Rogers' version one may clearly see the parallels with the Dunford-Schwartz proof, but a key step, the differentiation of an integral (its derivative is 0 and hence the integral is constant), is replaced by an argument to the effect that this integral is polynomial in time and constant near zero, hence is a constant. This argument totally removes identity (3) from the proof.

With the exception of Hadamard [Had], the authors of the cited papers don't give any indication of the source of the identity (3), either saying that it is well-known (as Heinz [Heinz] does), or, usually, just proving it. Hadamard mentions that this identity is well known in the theory of the multipliers. This classical theory is due to Jacobi [Jac] and is presented in the classical courses of Analysis, such as E. Goursat [G] or C. Jordan [Jor]. It generalizes the more well known Euler's theory of integrating factors (see, for example, [B-R] for the latter). The determinantal identity (3) is, in fact, the *Lemma fundamentale* of the Jacobi's paper; it is the first and the basic result of [Jac]. It turns out that the differential forms provide a very convenient framework for this theory. In the remaining part of the paper we will explain what are the multipliers (and how they can be used) and prove their existence using the differential forms language. The identity (3) plays a key role in the classical proofs of the existence of multipliers (and is the *raison d'être* for discussing the Jacobi multipliers here), and we will point out the corresponding part of our arguments.

Jacobi multipliers. Let X be a non-zero vector field in a domain in $\mathbf{R}^n$ with coordinates $(x_1, \ldots, x_n)$,

$$X = X_1 \frac{\partial}{\partial x_1} + \cdots + X_n \frac{\partial}{\partial x_n}.$$

A standard problem of theory of ordinary differential equation is to look for the integral curves of such a vector field X, i.e. for the curves in $\mathbf{R}^n$ such that their tangent lines have the same direction as the vectors of X at corresponding points. In particular, this problem really depends only on the line field generated by X. A satisfactory solution of this problem would be a family of $n-1$ functions $f_2, \ldots, f_n$ such that every integral curve is given by an equation of the form $f_2(x) = c_2, \ldots, f_n(x) = c_n$ for some constants $c_2, \ldots, c_n$. Of course, it is a basic theorem of the theory of ordinary differential equation that such a family of functions exists at least locally. The problem is how to find it for a given X. It turns out that a function μ such that

$$\frac{\partial \mu X_1}{\partial x_1} + \cdots + \frac{\partial \mu X_n}{\partial x_n} = 0 \qquad (4)$$

may be of some help. Such a μ is called a *multiplier* for X. If $n = 2$, then the multiplier is nothing else as the well known *integrating factor*. One may find an exposition of the Euler's theory of integrating factors in the already mentioned textbook [B-R], or in many other textbooks. While in the case $n = 2$ the knowledge of an integrating factor allows to find integral curves in the above sense, for $n > 2$ the knowledge of the multiplier helps only to find the last function f_n if $f_2, \ldots f_{n-1}$ are already known (note that for $n = 2$ there is only the last function $f_2 = f_n$).

The first problem of the theory is to prove that multipliers do exist. In order to do this, we will express the equation (4) in the language of differential forms.

Our main tools will be the Lie derivative L_Y along a vector field Y, the interior multiplication i_Y by Y and the Cartan homotopy formula

$$L_Y = d\, i_Y + i_Y d$$

relating them.

Let Ω be the standard volume form $dx_1 \wedge \ldots \wedge dx_n$ on $\mathbf{R}^n$. Now we will define a *multiplier* for X as a function μ such that $L_{\mu X}\Omega = 0$. In order to see that this definition is equivalent to the above classical one, let us compute $L_Y \Omega$ for a vector field

$$Y = Y_1 \frac{\partial}{\partial x_1} + \cdots + Y_n \frac{\partial}{\partial x_n}.$$

By the Leibniz formula for L_Y, we have

$$L_Y\Omega = L_Y(dx_1 \wedge \ldots \wedge dx_n) = \sum_{i=1}^{n} dx_1 \wedge \ldots \wedge L_Y dx_i \wedge \ldots \wedge dx_n.$$

Now, $L_Y dx_i = (di_Y + i_Y d)\, dx_i = di_Y dx_i + d(dx_i) = di_Y dx_i = dY_i$. If we substitute

$$dY_i = \frac{\partial Y_i}{\partial x_1} dx_1 + \cdots + \frac{\partial Y_i}{\partial x_n} dx_n$$

in the above formula for $L_Y\Omega$, we immediately get

$$L_Y\Omega = (\sum_{i=1}^{n} \frac{\partial Y_i}{\partial x_i})\Omega = (\frac{\partial Y_1}{\partial x_1} + \cdots + \frac{\partial Y_n}{\partial x_n})\Omega$$

It follows that $L_{\mu X}\Omega = 0$ if and only if

$$\frac{\partial \mu X_1}{\partial x_1} + \cdots + \frac{\partial \mu X_n}{\partial x_n} = 0,$$

and hence the two definitions are equivalent.

In order to prove the existence of multipliers we need a little piece of the multilinear algebra. Let V be a vector space of dimension n. Notice that it is dual to the space of 1-forms on V. If a volume form Ω on V is fixed, there is a natural nondegenerate pairing $\langle \cdot, \cdot \rangle$ between 1-forms and $(n-1)$-forms on V, given by

$$\omega_1 \wedge \omega_2 = \langle \omega_1, \omega_2 \rangle \Omega.$$

So, both the space of vectors and the space of $(n-1)$-forms are dual to the space of 1-forms, and hence there is a natural isomorphism between them. Let Y be the vector corresponding to a $(n-1)$-form A under this isomorphism. So, Y is defined by the requirement that equation $\omega(Y)\Omega = \omega \wedge A$ holds for all 1-forms ω on V. Another way to express this is by requiring the formula $(i_Y\omega)\Omega = \omega \wedge A$ to hold for all 1-forms ω, where i_Y is the interior multiplication by Y. By the Leibniz formula for the interior multiplication i_Y we have

$$i_Y(\omega \wedge \Omega) = i_Y\omega \wedge \Omega - \omega \wedge i_Y\Omega.$$

Since $\omega \wedge \Omega = 0$ as an $(n+1)$-form on an n-dimensional space, we conclude that $(i_Y\omega)\Omega = i_Y\omega \wedge \Omega = \omega \wedge i_Y\Omega$. Hence, the formula $(i_Y\omega)\Omega = \omega \wedge A$ is equivalent to $\omega \wedge i_Y\Omega = \omega \wedge A$. Clearly, the last formula holds for all ω if and only if $i_Y\Omega = A$. To sum up, Y corresponds to A under our isomorphism if and only if $i_Y\Omega = A$.

Of course, all this applies to vector fields and differential forms (fields of exterior forms). In what follows, Ω is again the standard volume form $dx_1 \wedge \ldots \wedge dx_n$. As we already noticed, there exist (at least locally) $n-1$ functions $f_1, \ldots, f_n$ such that every integral curve is given by an equation of the form $f_2(x) = c_2, \ldots, f_n(x) = c_n$. The functions $f_1, \ldots, f_n$ may be assumed independent in the sense that the $(n-1)$-form $A = df_2 \wedge \ldots \wedge df_n$ is nowhere zero. Consider the vector field Y corresponding to A under the above isomorphism, i.e. such that $i_Y \Omega = A$. Then

$$L_Y \Omega = (di_Y + i_Y d)\Omega = di_Y \Omega + i_Y d\Omega = di_Y \Omega = dA =$$

$$= d(df_2 \wedge \ldots \wedge df_n) = 0.$$

This proves that $L_Y \Omega = 0$. Note that in the final step we used the formula (2) (with f_i's in the role of φ_i's), which is equivalent to the identity (3). In the classical expositions [G], [Jor] the identity (3) itself shows up. Now we claim that the vector field Y is proportional to X. In fact, $\omega(Y)\Omega = \omega \wedge A$ for every 1-form ω by the definition of our isomorphism. In particular, $df_i(Y)\Omega = df_i \wedge A$. But $df_i \wedge A = df_i \wedge df_2 \wedge \ldots \wedge df_n = 0$ for all $i \geq 2$. It follows that $df_i(Y) = 0$ for all $i \geq 2$ and hence Y is tangent to the integral curves of X. This implies that Y is proportional to X, i.e. $Y = \mu X$ for some function μ. Obviously, we have $L_{\mu X}\Omega = L_Y \Omega = 0$. Hence, μ is a multiplier for X.

This proves the existence of multipliers. The reader may try to find the differential forms versions of the other parts of the theory as presented, for example, in [G], Sections 395, 396.

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Michigan State University
Department of Mathematics
Wells Hall
East Lansing, MI 48824-1027

E-mail: ivanov@math.msu.edu