Ten Problems on the Mapping Class Groups

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Let S be a compact orientable surface, possibly with a boundary. We denote by Mod_S the *mapping class group* of S, i.e. the group $\pi_0(\mathrm{Diff}(S))$ of isotopy classes of diffeomorphisms of S. This group is also known as the *Teichmüller modular group* of S, whence the notation Mod_S. Note that we include the isotopy classes of orientation-reversing diffeomorphisms in Mod_S, so our group Mod_S is what sometimes called the *extended* mapping class group.

For any property of discrete groups one may ask if Mod_S has this property. A better guidance is provided by the well-established by now analogy between the mapping class groups and arithmetic groups. See, for example, N. V. Ivanov, *Complexes of curves and the Teichmüller modular group,* Russian Math. Surveys, V. 42, No. 3 (1987), 55-107 for a discussion. In the following I tried to single out some specific questions, leaving aside such well-known problems as the existence of finitely dimensional faithful linear representations or the computation of the cohomology groups.

The original version of this paper was written in early 1994. Various updates (see below) are added approximately at the indicated times.†

1. The Congruence Subgroups Problem. Suppose that S is closed. Recall that a subgroup Γ of a group G is called *characteristic* if Γ is invariant under all automorphisms of G. If Γ is a characteristic subgroup of $\pi_1(S)$, then there is a natural homomorphism $\mathrm{Out}(\pi_1(S)) \to \mathrm{Out}(\pi_1(S)/\Gamma)$, where for a group G we denote by $\mathrm{Out}(G)$ the quotient of the group $\mathrm{Aut}(G)$ of all automorphisms of G by the (automatically normal) subgroup of all inner automorphisms. Clearly, if Γ is of finite index in $\pi_1(S)$, then the kernel of this homomorphism is also of finite index. Note that any subgroup of finite index in a finitely generated group (in particular, in $\pi_1(S)$) contains a characteristic subgroup of finite index. Since by the Dehn-Nielsen theorem Mod_S is canonically isomorphic to $\mathrm{Out}(\pi_1(S))$,

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*The 2014 version differs from the previous 1998 version only in use of a new typesetting style and adding four footnotes, including this one. No attempt to update the text or to systematically correct misprints was made.

†I was adding updates about the current status of the problems till the Fall of 1998. For further updates, see my paper *Fifteen Problems about the Mapping Class Groups* – Added on January 7, 2014.

our construction gives rise to a family of subgroups of finite index in Mod_S. By analogy with the classical arithmetic groups we call them the *congruence subgroups*.

Conjecture. *Every subgroup of finite index in* Mod_S *contains a congruence subgroup.*

V. Voevodsky had indicated (in a personal communication) a beautiful application of this conjecture. Namely, the conjecture implies that a smooth algebraic curve over $\mathbf{Q}$ is determined up to an isomorphism by its algebraic fundamental group (which is isomorphic to the profinite completion of $\pi_1(S)$) considered together with the natural action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on it.

Update (Spring of 1996). This corollary (which was apparently first conjectured by A. Grothendieck) of the conjecture was proved by S. Mochizuki and A. Tamagawa of RIMS, Kyoto University by algebro-geometric methods. This lends some additional credibility to the conjecture.

Update (Fall of 1997). The results mentioned in the last update are published recently. See A. Tamagawa, *The Grothendieck conjecture for affine curves*, Compositio Mathematica, V. 109 (1997), 135-194, and Sh. Mochizuki, *The profinite Grothendieck conjecture for closed hyperbolic curves over number fields*, J. Math. Sci. Univ. Tokyo, V. 3 (1996), 571-627. For further spectacular results in this direction, see Sh. Mochizuki, *The local pro-*p *anabelian geometry of curves*, Preprint RIMS-1097, Kyoto (1996), 84 pp. and references therein.

2. Normal Subgroups. If a subgroup Γ of $\pi_1(S)$ is characteristic, then the kernel of the natural homomorphism $\mathrm{Out}(\pi_1(S)) \to \mathrm{Out}(\pi_1(S)/\Gamma)$ from the Problem 1 is a normal subgroup of $\mathrm{Out}(\pi_1(S))$. So, this construction gives rise to a family of normal subgroups of Mod_S. In general, these subgroups have infinite index. For example, the Torelli subgroup is clearly of this type.

Question. *Is it true that any normal subgroup is commensurable with such a subgroup?*

Recall that two subgroups Γ_1, Γ_2 of a group G are *commensurable* if the intersection $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2.)

This problem was suggested by a discussion with H. Bass.

3. Normal Subgroups and pseudo-Anosov Elements (D. D. Long, J. D. McCarthy, R. C. Penner). *Is it possible that all nontrivial (i.e.,* $\neq 1$*) elements of a* normal *subgroup of* Mod_S *are pseudo-Anosov?*

4. Conjecture (Mostow-Margulis Super-rigidity). *If Γ is an irreducible arithmetic group of rank $\geqslant 2$, then every homomorphism $\Gamma \to \mathrm{Mod}_S$ has finite image.*

For many arithmetic groups Γ the conjecture can be proved by combining some well-known information about arithmetic groups with equally well-known properties of Mod_S. But, for example, for cocompact lattices in $SU(p, q)$ this straightforward approach seems to fail. (I owe this specific example to G. Prasad.)

Update (Spring of 1995). Recently, V. A. Kaimanovich and H. Masur proved that so-called non-elementary subgroups of Mod_S are not isomorphic to irreducible arithmetic groups of rank $\geqslant 2$. The proof is based on the theory of random walks on Mod_S developed by Masur and Kaimanovich–Masur and on the results of H. Furstenberg about random walks on arithmetic groups. See V. A. Kaimanovich and H. Masur, *The Poisson boundary of the mapping class group*, Inventiones Math., V. 125, F. 2 (1996), 221-264. If combined with the Margulis finiteness theorem and the technique of N. V. Ivanov, *Subgroups of Teichmüller Modular Groups*, Translations of Mathematical Monographs, Vol. 115, American Math. Soc., 1992, this result easily implies the conjecture.

Update (Summer of 1996). Recently I found a more direct proof of this conjecture. It follows in the outline the approach of Margulis to super-rigidity and uses only some elementary (but very crucial) results of the paper of Kaimanovich and Masur cited above.

Update (Summer of 1997 and Fall of 1998). For a proof close in the spirit to the one mentioned in the first update, see B. Farb and H. Masur, *Superrigidity and mapping class groups*, Topology, V. 37, No. 6 (1998), 1169-1176.

5. Dehn Multi-twists. For a nontrivial circle α on S let us denote by t_α the (left) Dehn twist along α; $t_\alpha \in \mathrm{Mod}_S$. A *Dehn multi-twist* is any composition of the form $t_{\alpha_1}^{\pm 1} \circ t_{\alpha_2}^{\pm 1} \circ \cdots \circ t_{\alpha_n}^{\pm 1}$ for disjoint circles $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Question. *Is there a constant N_S, depending only on S, such that the following holds? Let $f \in \mathrm{Mod}_S$ and $t = t_{\alpha_1}^{\pm 1} \circ t_{\alpha_2}^{\pm 1} \circ \cdots \circ t_{\alpha_n}^{\pm 1}$ is a Dehn multi-twist. If $A = \{f^m(\alpha_i) : 1 \leqslant i \leqslant n, m \in \mathbf{Z}\}$ fills S (i.e. for any nontrivial circle γ there exist an $\alpha \in A$ such that $i(\gamma, \alpha) \neq 0$), then only a finite number of elements $t^j \circ f$, $j \in \mathbf{Z}$ are not pseudo-Anosov, and they are among N_S consecutive elements of this family.*

By a theorem of A. Fathi, *Dehn twists and pseudo-Anosov diffeomorphism*, Inventiones Math., V. 87, No. 1 (1987), 129-151, if t is a Dehn *twist*, then this is true for $N_S = 7$.

A weaker version of this question is still interesting: under the same conditions, is it true that no more than N_S elements among $t^j \circ f$ are not pseudo-Anosov? A positive answer to this question would allow to prove the Conjecture 4 in some nontrivial cases.

6. Automorphisms of Complexes of Curves. The *complex of curves* $C(S)$ of a surface S is a simplicial complex defined as follows. The vertices of $C(S)$ are the isotopy classes of nontrivial (i.e., not bounding a disk and non deformable into the boundary) circles on S. A set of vertices forms a simplex if and only if they can be represented by disjoint circles. This notion was introduced by W. J. Harvey, *Boundary structure of the modular group*, in: *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook conference*, Ed. by I. Kra and B. Maskit, Annals of Mathematics Studies, No. 97 (1981), 245-251. Clearly, Mod_S acts on $C(S)$, and this action is almost always effective (the main exception is the case of a closed surface of genus 2, when the hyperelliptic involution acts trivially). If the genus of S is at least 2, then it is known (N. V. Ivanov, *Automorphisms of complexes of curves and of Teichmüller spaces,* Preprint IHES/M/89/60) that all automorphisms of $C(S)$ come from Mod_S.

Question. *Is the same true for surfaces of genus* 1 *and* 0*?*

Update (Fall of 1994). M. Korkmaz proved that all automorphisms of $C(S)$ come from Mod_S for all surfaces of genus 0 and 1 with the exception of spheres with $\leqslant 4$ holes and tori with $\leqslant 2$ holes. These results are included in M. Korkmaz, *Complexes of curves and mapping class groups*, Ph. D. Thesis, Michigan State University, 1996. It is available on my home page‡ (cf. http://www.mth.msu.edu/~ivanov/).

Update (Spring of 1997). F. Luo suggested a new proof of the above results about automorphisms of $C(S)$ and also observed that $\mathrm{Aut}(C(S))$ is not equal to Mod_S if S is a torus with 2 holes. The reason is very simple: if $S_{1,2}$ is a torus with 2 holes, and $S_{0,5}$ is a sphere with 5 holes, then $C(S_{1,2})$ is isomorphic to $C(S_{0,5})$, but $\mathrm{Mod}_{S_{1,2}}$ is not isomorphic to $\mathrm{Mod}_{S_{0,5}}$. See F. Luo, *Automorphisms of complexes of curves,* Preprint, 1997.

Update (Summer of 1997). For a detailed discussions of these and related topics, see N. V. Ivanov, *Automorphisms of complexes of curves and of Teichmüller spaces,* International Mathematics Research Notices, 1997, No. 14, 651-666.

7. The First Cohomology Group of Subgroups of Finite Index. It is well known that $H^1(\mathrm{Mod}_S) = 0$.

Question. *Is it true that* $H^1(\Gamma) = 0$ *for any subgroup* Γ *of finite index in* Mod_S*?*

Update (Summer of 1995). F. Taherkhani (a graduate student at MSU) proved that $H_1(\Gamma_2(2)) = \mathbf{Z}^9 \oplus (\mathbf{Z}/2\mathbf{Z})^4 \oplus (\mathbf{Z}/4\mathbf{Z})$, where $\Gamma_2(2)$ is the intersection of the subgroup of orientation-preserving isotopy classes with the kernel of the natural homomorphism $\mathrm{Mod}_{S_2} \to$

‡Not anymore. – Added January 7, 2014.

Aut $(H_1(S_2, \mathbf{Z}/2\mathbf{Z}))$, and where S_2 is a closed surface of genus 2. In particular, the subgroup $\Gamma_2(2)$ of Mod_{S_2}, which is obviously of finite index, has infinite first homology (and hence first cohomology) group. The methods of Taherkhani are based on extensive computer calculations.

J. D. McCarthy proved that if S is a closed surface of genus $\geqslant 3$ and Γ is a subgroup of finite index in Mod_S containing the Torelli subgroup, then the first cohomology group $H^1(\Gamma)$ is trivial. His methods are based on D. Johnson results about the Torelli subgroup, the solution of the congruence subgroups problem for $Sp_{2g}(\mathbf{Z})$, $g \geqslant 3$, and the Kazhdan property T of $Sp_{2g}(\mathbf{Z})$, $g \geqslant 3$.

Update (Summer of 1997). Recently, F. Taherkhani found a subgroup of finite index in Mod_S with nontrivial first cohomology group for a closed surface S of genus 3. The proof is again based on extensive calculations with the aid of a computer. See F. Taherkhani, *The Kazhdan property of the mapping class group of closed surfaces and the first cohomology group of their cofinite subgroups*, Preprint, 1997, 25 pp.

The results of J. D. McCarthy mentioned in the previous update are now available in J. D. McCarthy, *On the first cohomology group of cofinite subgroups in surface mapping class groups,* Preprint, 1997, 19 pp.

Update (Fall of 1998). The subgroup mentioned in the previous update turned out to have the first cohomology group equal to 0 after a reexamination (done by F. Taherkhani) of the calculations.

8. Kazhdan Property T. *Does* Mod_S *has the Kazhdan property T?*

The positive answer will imply a positive answer to the previous question, but this problems seems to be much more difficult. In fact, all known proofs of the property T for discrete groups are eventually based on their relations with Lie groups and on the representation theory of Lie groups. Such an approach is not available for Mod_S.

Update (Summer of 1995). It follows from the new results of F. Taherkhani mentioned above that mapping class groups of surfaces of genus 2 do not have the property T.

Update (Summer of 1997). It follows from the results of F. Taherkhani mentioned above that mapping class groups of surfaces of genus 3 do not have the property T either. This strongly suggests that all mapping class groups do not have the property T.

Update (Fall of 1998). In view of the last update of the Problem 7, the question about the property T of the mapping class groups of surfaces of genus 3 remains open.

9. Unipotent Elements. Let $d_W(\cdot,\cdot)$ be the word metric on Mod_S with respect to some finite set of generators. Let $t \in \mathrm{Mod}_S$ be a Dehn twist. What is the growth rate of $d_W(t^n, 1)$? One may expect that either the growth is linear, or $d_W(t^n, 1) = O(\log n)$. In the arithmetic groups case, logarithmic growth corresponds to virtually unipotent elements of arithmetic groups of rank $\geqslant 2$, according to a recent theorem of A. Lubotzky, S. Moses and M.S. Raghunathan, *Cyclic subgroups of exponential growth and metrics on discrete groups*, C. R. Acad. Sci. Paris, t. 317, Série I (1993), 735-740.

Update (Summer of 1997). Y. Minsky proved that the growth is linear. See Y. Minsky, *Dehn twists have linear growth,* Preprint, 1997, 5 pp.

10. Nonorientable Surfaces. Most of the work on the mapping class groups is done for orientable surfaces only. Some exceptions are the paper of M. Scharlemann, *The complex of curves of a nonorientable surface*, J. London Math. Soc., V. 25 (1982), 171-184, and the computation of the virtual cohomology dimension for the mapping class groups of non-orientable surfaces in N. V. Ivanov, *Complexes of curves and the Teichmüller modular group,* Russian Math. Surveys, V. 42, No. 3 (1987), 55-107. One may look for analogues of other theorems about Mod_S for nonorientable surfaces. For example, what are the automorphisms of mapping class groups of nonorientable surfaces?

Update (Fall of 1995 and Fall of 1998). Some work about mapping class groups of nonorientable surfaces was done recently by M. Korkmaz. See his 1996 MSU Ph. D. Thesis, cited above, and M. Korkmaz, *First homology group of mapping class group of nonorientable surfaces*, Mathematical Proc. Cambridge Phil. Soc., V. 123, No. 3 (1998), 487-499. See also M. Korkmaz, *On generators of the mapping class group of a nonorientable surface,* Preprint. His thesis and the latter paper are available at my home page[§] (cf. http://www.mth.msu.edu/~ivanov/).

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[§]Not anymore. – Added January 7, 2014.